Math 561, Fall 2018
Assignment 12, due Friday, November 30

Hand in solutions to the following exercises:

1. (Greene & Krantz, Chapter 5, Exercise 8, rewritten) Let $f : U \to \mathbb{C}$ be holomorphic. Assume that $\overline{D}(P, r+\delta) \subset U$ for some $r, \delta > 0$ and that f is nonvanishing on $\partial D(P, r)$. Show there exists $\epsilon > 0$ depending only on f, P, r so that if g is holomorphic on U and

$$\sup_{\zeta \in \overline{D}(P, r+\delta)} |f(\zeta) - g(\zeta)| < \epsilon,$$

then f and g have the same number of zeros in $D(P, r)$, counting multiplicities.

Note: It may be helpful to set $M = \sup_{\zeta \in \overline{D}(P,r+\delta)} |f(\zeta)|$, $m = \inf_{\zeta \in \partial D(P,r)} |f(\zeta)|$, observing that $m > 0$, so that $\sup_{\zeta \in \overline{D}(P,r+\delta)} |g(\zeta)| < M + \epsilon$, $\inf_{\zeta \in \partial D(P,r)} |f(\zeta)| > m - \epsilon$. You will likely want to restrict attention to $\epsilon \leq m/2$, or another small fraction of m (why?). The role of the δ in the above is so that the Cauchy estimates give upper bounds on $\sup_{\zeta \in \partial D(P,r)} |f'(\zeta) - g'(\zeta)|$ and $\sup_{\zeta \in \partial D(P,r)} |f'(\zeta)|$. See also the very similar Exercise 1 in the "On your own" section below.

2. Show that the polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has all of its roots in the disc $D(0, R)$ where $R = 1 + \max\{|a_0|, \ldots, |a_{n-1}|\}$.

3. Suppose f is continuous on $\overline{D}(0, 1)$, holomorphic on $D(0, 1)$, and satisfies $|f(z)| \geq \sqrt{|z|}$ for all $|z| \leq 1$. Prove that $|f(0)| \geq 1$.

   Hint: Apply the maximum principle to $\frac{1}{f(z)}$. The catch is that nothing says that $f(0) \neq 0$ to begin with, so you have show that $\frac{1}{f(z)}$ has a removable singularity at 0.

4. Suppose U is a *bounded domain* (i.e. a connected open set which is bounded). Suppose that f is continuous on $\overline{U}$ and holomorphic on U. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all $z \in \partial U$, then either f has a zero in U or f is a constant function.

On your own: Greene & Krantz: Ch. 5, Exercises 6[1], 10, 11, 14[2]. Also, the following exercise:

1. Suppose $f_j, g_j : X \to \mathbb{C}$ are sequences of functions on a set X converging uniformly to *bounded* limit functions f, g respectively. Show that the products $f_j \cdot g_j$ converge uniformly to $f \cdot g$. Show that if the g_j are nonvanishing on X and the limit function g satisfies $|g(x)| \geq m > 0$ for all $x \in X$, then the quotients f_j/g_j converge to f/g uniformly on X.

Reading: Greene & Krantz, finish Chapter 5, start Chapter 6.

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[1] A holomorphic logarithm refers to a holomorphic function g satisfying $e^{g(z)} \equiv f(z)$. Use what you know about holomorphic antiderivatives from Chapter 1.

[2] Don't hesitate to use Exercise 1 (or Exercise 8 in Ch.5) from above here. Recall that a continuous function on a compact set is always uniformly continuous.