Math 561, Fall 2018
Assignment 11, due Wednesday, November 21

Hand in solutions to the following exercises:

1. (a) Prove that if $\alpha \in \mathbb{C}$ is not an integer on the real axis, then

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+\alpha)^2} = \frac{\pi^2}{\sin^2 \pi\alpha}$$

   (b) Manipulate the identity in part (a) and take limits as $\alpha \to 0$ to give a proof that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

2. Suppose f is holomorphic on an open set U containing $\{z : |z| > R\}$ for some $R > 0$ so that f is given by a convergent Laurent series $f(z) = \sum_{j=-\infty}^{\infty} a_j z^j$ on $\{z : |z| > R\}$. Define the *residue at infinity* of f as $\mathrm{Res}_f(\infty) = -a_{-1}$, where a_{-1} is the coefficient of z^{-1} in the Laurent expansion.

   (a) Since f is holomorphic on $\{z : |z| > R\}$, the function $H(z) = z^{-2} f(1/z)$ is holomorphic on $D^*(0, 1/R)$. Prove that $\mathrm{Res}_f(\infty) = -\mathrm{Res}_H(0)$.

   (b) Suppose $s > R$. Apply Proposition 4.2.4 to the function H and perform a change of variables to show that

$$\mathrm{Res}_f(\infty) = \frac{1}{2\pi i} \oint_{|z|=s} f(z)\, dz.$$

   where the line integral is oriented *clockwise.*

   Note: Be careful with the change of variables, limit yourself to the change of variables theorem for functions of a single variable ("u-substitution").

3. Suppose that f is an entire function for which there exists $M, R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \geq M|z|^n$. Prove that f is a polynomial of degree at least n.

4. Greene & Krantz, Chapter 5, Exercise 2.

   Note: You should be able to express your answer in terms of $g(P_1), \ldots, g(P_k)$. There is nothing special about taking the disc to be centered at 0 with radius 1, so treat this only as a convenience and convince yourself that this works more broadly.

5. Greene & Krantz, Chapter 5, Exercise 3.

On your own: Greene & Krantz: Ch. 4, Exercises 18, 28, 59, 60, 61, 65. Ch. 5, Exercises 1, 4, 8. Also, the following exercises:

1. Prove that if $w \in \mathbb{C}$ is not an integer on the real axis, then $\pi \cot \pi w = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{n+w}$.

2. (a) Suppose f is bounded and holomorphic except at a finite number of isolated singularities in $\mathbb{C}$. Prove that f is constant.

   (b) Let f be holomorphic except at a finite number of poles and suppose that for some $M, R > 0$ and $n \in \mathbb{N}$, $|f(z)| \leq M|z|^n$ whenever $|z| > R$. Prove that f must be a rational function.

Reading: Greene & Krantz: finish Chapter 4, start Chapter 5, namely §5.1.