Math 561, Fall 2018
Assignment 10, due Wednesday, November 7
NOT COLLECTED

On your own:

1. Suppose $f : D^*(P, r_0) \to \mathbb{C}$ has a simple pole at P. For $0 < r < r_0$, define $\gamma_r(t) = P + re^{it}$ over the domain $[\alpha, \alpha+\theta]$ where $\alpha \in \mathbb{R}$ and $\theta > 0$. Prove by direct computation that
$$\lim_{r \to 0+} \oint_{\gamma_r} f(z)\, dz = i\theta \operatorname{Res}_f(P).$$
Hint: Begin by observing that $f(z) = \frac{\operatorname{Res}_f(P)}{z-P} + g(z)$ on $D^*(P, r_0)$ for some $g(z)$ with a removable singularity at P.

2. Suppose that $f : D^*(0, r_0) \to \mathbb{C}$ has an isolated singularity at P and that $f(z) = f(-z)$ for all $z \in D^*(0, r_0)$. Prove that $\operatorname{Res}_f(0) = 0$.

3. Use the calculus of residues to compute the following integrals:
   (a) $\int_0^\infty \frac{1}{(1+x^2)^2}\, dx$.
   (b) $\int_0^\infty \frac{1}{1+x^{2n}}\, dx = \frac{\pi/2n}{\sin(\pi/2n)}$.
   Hint: Consider what happens to f along the rays $\{t : t > 0\}$ and $\{te^{i\pi/n} : t > 0\}$.
   (c) $\int_{-\infty}^\infty \frac{x}{\sinh x}\, dx$.
   Hint: What happens if you consider rectangular contours with 2 parallel sides on $\operatorname{Im}(z) = 0$ and $\operatorname{Im}(z) = \pi i$? This actually won't work, because $z/\sinh z$ has a pole at πi, but there are ways around this. The result in Problem 1 above may be helpful to this end. Note that the singularity of $z/\sinh z$ at 0 is removable.
   (d) $\int_0^\infty \frac{1}{p(x)}\, dx$, where $p(x)$ is a polynomial with no zeros on the nonnegative real axis.
   Note: Express your answer in terms of the residues of $1/p(z)$. But give a more explicit answer in the special case where all zeros of $p(z)$ are simple and the result in Exercise 38 applies. If you follow the method outlined in Example 4.6.5, give proofs that the limit of the integrals over η_R^2, η_R^4 vanishes as $R \to \infty$.
   (e) $\int_{-\infty}^\infty \frac{e^{bx}}{e^{ax}+1} = \frac{\pi}{a\sin(\pi b/a)}$, where $a > b > 0$.
   Hint: Try a rectangular contour of height $2\pi/a$.

Reading: Greene & Krantz: finish Chapter 4.