Math 402/502, Spring 2020
Assignment 6, due Thursday, March 12

Problems to hand in:

1. Let $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ be the function considered in Exercise 7.2.3. Use Corollary 7.34 and the binomial formula to show that $E(x)E(y) = E(x+y)$ for each $x, y \in \mathbb{R}$.

   Note: As shown in Example 7.45, $E(x) = e^x$ and this is of course a familiar property of exponents. The goal of the problem is therefore to derive the same property independently using results on multiplication of power series.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$
   Prove that f has derivatives of all orders on $\mathbb{R}$ with $f^{(n)}(0) = 0$ for all n. You may use all usual properties of the exponential functions, including $\frac{d}{du}e^u = e^u$ and $\lim_{u\to\infty} e^u = \infty$, $\lim_{u\to-\infty} e^u = 0$,

   Hint: first prove that for every $n \in \mathbb{N}$, the function $f^{(n)}(x)$ on $(0, \infty)$ takes the form $p_{3n}(\frac{1}{x})e^{-1/x^2}$ for some p_k is a polynomial of degree k. Then show $\lim_{x\to 0+} \frac{1}{x^k}e^{-1/x^2} = 0$ for every $k \in \mathbb{N}$.

   Note: This is a slight modification of the function in Remark 7.41. The version in this exercise arises in some applications, such as the construction of "bump functions". It is an example of a $\mathcal{C}^\infty$ function which is not equal to its Taylor series.

3. Wade, Exercise 10.1.9.

4. Define the following for $x, y \in \mathbb{R}$:
$$\begin{aligned} d_1(x,y) &:= (x-y)^2, \\ d_2(x,y) &:= \sqrt{|x-y|}, \\ d_3(x,y) &:= |x^2 - y^2|, \\ d_4(x,y) &:= |x - 2y|. \end{aligned}$$
   Which of these define a metric on $\mathbb{R}$ and which do not? Fully justify your answers.

On your own: Wade 7.2.3, 10.1.3, as well as the following problem:

1. Let $\mathcal{C}[a,b]$ be the set continuous functions on $[a,b]$, and let ρ denote the metric on $\mathcal{C}[a,b]$ defined in Example 10.6 of the text: $\rho(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathcal{C}[a,b]$. Show that $f_n \to f$ converges uniformly as $n \to \infty$ if and only if $\{f_n\}_{n=1}^{\infty}$ is a convergent sequence in this metric space.

Reading: Section 10.1 of the Wade text. Section 7.4 is also worthwhile for your edification. You also should review §8.1, 8.2 in the Wade text concerning the fundamentals of n-dimensional Euclidean space (i.e. $\mathbb{R}^n$). This should be a fairly straightforward review, but this is relevant as we proceed in the course, so make sure to spend time on anything which is new to you. We will cover most of §8.3 and §8.4 in the more general setting of metric spaces as we proceed through Chapter 10.