Math 402/502, Spring 2020
Assignment 5, due Wednesday, February 27

Problems to hand in:

1. Decide if the following sequences of functions converge uniformly, pointwise but not uniformly, or neither on the given set E. Justify your answers.
   (a) $f_n(x) = x/n$, $E = (0, 1)$.
   (b) $f_n(x) = \sin(x/n)$, $E = \mathbb{R}$.
   (c) $f_n(x) = \sin(n\pi x)$, $E = \mathbb{Q}$.
   (d) $f_n(x) = x^2 + \frac{x}{n}$, $E = [0, \infty)$.

2. (a) Wade, Exercise 7.1.3.
   (b) Suppose f_n, g_n are sequences of functions on a set $E \subset \mathbb{R}$. Show that if f_n is a uniformly bounded sequence and $g_n \to 0$ uniformly on E, then $f_n g_n \to 0$ on E.

3. Wade, Exercise 7.1.7.

4. Wade, Exercise 7.2.5.

   Hint: You may of course assume all the usual properties of sine and cosine from calculus. Begin by proving that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$, either the mean value theorem or the fundamental theorem of calculus are effective to this end. You may also find it helpful to use that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

On your own: Wade 7.1.1, 7.1.6, 7.2.2 as well as the following problems:

1. Consider the following sequences of functions on the given domain E in the context of Exercise 1 above:
   (a) $f_n(x) = x^n(1 - x)$, $E = [0, 1]$. Hint: Find the maximum of $x^n(1 - x)$ on $[0, 1]$.
   (b) $f_n(x) = \sin(x/n)$, $E = [-b, b]$ with $b > 0$.

2. Suppose $f_n \to f$ and $g_n \to g$ uniformly on a set $E \subset \mathbb{R}$ and let $\alpha \in \mathbb{R}$ be a constant. Show that $f_n + g_n \to f + g$ uniformly and $\alpha f_n \to \alpha f$ uniformly on E.

3. Show that the series $\sum_{k=0}^{\infty} \frac{x^2}{(1+x^2)^k}$ converges pointwise to 0 if $x = 0$ and to $1 + x^2$ if $x \neq 0$. Is the convergence uniform on $[-1, 1]$? Is the convergence uniform on $[b, \infty)$ if $b > 0$? Justify your answers.

Reading: Wade, sections 7.1-7.4.