Math 402/502, Spring 2020
Assignment 4, due Thursday, February 20

Problems to hand in:

1. Wade, Exercise 6.3.7.

   Hint: For (b), it may be more palatable to prove the contrapositive statement "If $\sum_{k=1}^{\infty} k^p a_k$ converges for some $p > 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely."

2. Let $s_n = 1 + \frac{1}{2} + \cdots \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$. Prove that $s_{2k} - s_k > \frac{1}{2}$ and use this to show that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Do not appeal to the integral test.

3. Suppose $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ are bounded sequences of nonnegative real numbers. Show that if $\lim_{k\to\infty} x_k = L$ exists and $L > 0$, then

$$\limsup_{k\to\infty} x_k y_k = L \cdot \limsup_{k\to\infty} y_k.$$

   Note: A standard 401 exercise is to show that if instead $L = 0$, then we have the stronger conclusion that $\lim_{k\to\infty} x_k y_k = 0$ since $\{y_k\}$ is bounded. On your own, remind yourself how to do this if you've forgotten.

4. Let a_k be defined as $a_k = 2^{-k}$ for k odd and $a_k = 3^{-k}$ for k even. Show that

$$\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} < 1 < \limsup_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

   Conclude that the ratio test fails to give any information on the convergence or divergence of the series while the root test shows that $\sum_{k=1}^{\infty} a_k$ converges.

5. Suppose that $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ are real sequences such that $\sum_{k=1}^{\infty} |b_{k+1} - b_k|$ is convergent, $\lim_{k\to\infty} b_k = 0$, and there exists M such that $|\sum_{k=1}^{n} a_k| \leq M$ for any $n \in \mathbb{N}$. Prove that $\sum_{k=1}^{\infty} a_k b_k$ is convergent.

   Hint: Since $\{b_k\}$ is not decreasing, Dirichlet's test does not apply directly. Instead, use the Cauchy criteria for series and the Abel formula (showing that $|A_{n,m}| \leq 2M$).

On your own: Wade 6.2.6 (after reviewing the limit comparison test) and the following problems (see the back page as well):

1. (a) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

   (b) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

2. A *rational function* is a function of the form $f(x) = p(x)/q(x)$ defined for $x \in \mathbb{R}$ where $p(x) = \sum_{j=0}^{k} \alpha_j x^j$, $q(x) = \sum_{l=0}^{m} \beta_l x^l$ are both polynomials. Suppose $a_n = p(n)/q(n)$ where $p(n) \neq 0$ and $q(n) \neq 0$ for every n. Prove that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{p(n+1)}{p(n)} \cdot \frac{q(n)}{q(n+1)} \right| = 1,$$

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{p(n)}{q(n)} \right|^{1/n} = 1.$$

Conclude the ratio test *always* fails to give information when a_n is a rational function.

Note: This exercise shows there is a broad class of series for which the root and ratio tests fail to give information, e.g. $\sum \frac{n^3}{2n^5-1}$, $\sum \frac{n+1}{n^2-2}$, etc. So in these cases, you must use another test to deduce convergence or divergence of the series.

Reading: Wade, finish your review of series and Chapter 6. Start Ch. 7.