Math 402/502, Spring 2020
Assignment 3, due Thursday, February 13

Problems to hand in:

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function which is integrable on $[c, b]$ for each $c \in (a, b)$. Prove that f is integrable on $[a, b]$ and that $\int_a^b f(x)\,dx = \lim_{c\to a+} \int_c^b f(x)\,dx$.

   Note: This is reminiscent of Exercise 5.1.3 in the text; on your own, observe that it actually follows from the exercise here.

2. Wade, Exercise 5.2.6.

3. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function, i.e f is differentiable and f' is a continuous function. Use the first mean value theorem for integrals (Theorem 5.24) to derive the usual mean value theorem for derivatives, namely that there exists $c \in [a, b]$ such that
$$f(b) - f(a) = f'(c)(b - a).$$
   Note: Clearly your solution must involve Theorem 5.24, it should not appeal to the usual proofs of the mean value theorem you may have encountered in Math 401/501. It should be noted that the exercise here furnishes a weaker result than the usual mean value theorem (Theorem 4.15(ii)), as the latter does not assume that f is *continuously* differentiable, just merely differentiable.

4. Wade, Exercise 5.3.7.

   Note: The purpose of this exercise is to *define* the natural logarithm from principles in calculus, then derive some of the key properties of this function from the definition. For example, even though it is well-known that $\log(xy) = \log x + \log y$, in (d) you are to derive that property from the given definition, using a change of variables to justify
$$\int_1^{xy} \frac{dt}{t} = \int_1^{x} \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^{x} \frac{dt}{t} + \int_1^{y} \frac{dt}{t}.$$

5. Let $I \subset \mathbb{R}$ be an interval and fix $a \in I$. Suppose that $f : I \to \mathbb{R}$ is $n + 1$-times continuously differentiable, that is, $f^{(k)}(x)$ (the k-th derivative of $f(x)$) exists for all $1 \le k \le n+1$ and defines a continuous function. Prove *Taylor's theorem with integral remainder*: for each $x \in I$, $f(x)$ satisfies
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$
   with $R_n(x)$ given by
$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$
   Hint: The statement is amenable to an induction argument, and the $n = 0$ case is just the Fundamental Theorem of Calculus. Use integration by parts and that
$$-\frac{1}{n!}\frac{d}{dt}(x-t)^n = \frac{1}{(n-1)!}(x-t)^{n-1}.$$

On your own (i.e. do not hand these in for a grade): Wade 5.2.10 (along with 3.1.8 if needed), 5.4.2(a,b) as well as the following problem:

Using the conventions $\int_a^a f(x)\,dx = 0$ and $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ when $b < a$, show that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

regardless of the ordering of a, b, c *in the real number line.* Note that by elementary combinatorics, there are $3! = 6$ possible choices for the ordering of a, b, c (e.g. $a < b < c,\ a < c < b,\ b < a < c, \dots$).

Reading: Wade, finish your reading of §5.1-5.3, review the definition of improper integral in §5.4.