Math 402/502, Spring 2020
Assignment 2, due Thursday, February 6

Problems to hand in:

1. Wade, Exercise 5.1.3.
2. Wade, Exercise 5.1.4.
3. Wade, Exercise 5.1.8.
4. Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded functions and that there exists a uniform constant $C > 0$ such that $|f(x) - f(y)| \leq C|g(x) - g(y)|$ for all $x, y \in [a, b]$. Show that if g is integrable, then f is also integrable.

   Hint: Use the following identity from the "on your own" section in the previous homework: Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$: $M_j(f) - m_j(f) = \sup\{|f(t) - f(s)| : t, s \in [x_{j-1}, x_j]\}$.

On your own (i.e. do not hand these in for a grade): Wade 5.1.5, 5.1.10[1], as well as the following problems:

1. Suppose $g : [a, b] \to \mathbb{R}$ is integrable. Use Exercise #4 above to show that the following functions are also integrable on $[a, b]$
   (a) g^2
   (b) $\sqrt{g}$, with the additional assumption that there is a uniform constant $c > 0$ such that $g(x) \geq c$ for all $x \in [a, b]$. (Begin by showing that $\sqrt{g(x)} - \sqrt{g(y)} = \frac{g(x)-g(y)}{\sqrt{g(x)}+\sqrt{g(y)}}$).
2. Let $f : [a, b] \to \mathbb{R}$ be a function and $\{y_1, \ldots, y_n\} \subset [a, b]$ a finite collection of points in $[a, b]$. Suppose $f(x) = 0$ for each $x \in [a, b] \setminus \{y_1, \ldots, y_n\}$, that is, f vanishes except at finitely many points in $[a, b]$. Prove that f is integrable on $[a, b]$ and that $\int_a^b f(x)\,dx = 0$.

   Conclude that given this exercise and Theorem 5.19 in the text (linear property of the integral) that Exercise 5.1.6 in the text follows easily.
3. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function such that $m \leq f(x) \leq M$ for every $x \in [a, b]$ (i.e. m, M are lower and upper bounds for the range of f respectively). Prove that for any partition P,

   $$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a).$$

   Conclude that the sets $\{L(f, P) : P \text{ is a partition}\}$ and $\{U(f, P) : P \text{ is a partition}\}$ are bounded sets in $\mathbb{R}$. Note that this ensures that the upper and lower integrals $U(f)$ and $L(f)$ are well-defined to begin with!

Reading: Wade, Sections 5.1, 5.2, start 5.3.

[1] The absolute values appearing in $|U(f, P) - L(f, P)| < \varepsilon$ here aren't really necessary since the difference is always nonnegative for any partition P (see Remark 5.6).