Math 402/502, Spring 2020
Assignment 1, due Thursday, January 30

Problems to hand in

1. Suppose $x, y \in \mathbb{R}$. Use a proof by contrapositive to show the following statement:

   If $x \leq y + \epsilon$ for every $\epsilon > 0$, then $x \leq y$.

   Note: the contrapositive isn't at all hard to show here, but like the "on your own" problem below, this principle comes up **frequently**. We will be presented with numerous situations where it is prohibitive or simply inefficient to show $x \leq y$ directly, but approximation arguments will allow us to see $x \leq y + \epsilon$ for arbitrary choices of $\epsilon > 0$.

2. Suppose $A \subset \mathbb{R}$ is a nonempty set which is bounded from above and below. Prove that

$$\sup A - \inf A = \sup\{|a_1 - a_2| : a_1, a_2 \in A\} = \sup\{a_1 - a_2 : a_1, a_2 \in A\}.$$

3. Let $f(x) = \frac{x}{x+1}$, which is well-defined and continuous on any subset of $\mathbb{R} \setminus \{-1\}$.

   (a) Use the $\epsilon - \delta$ definition to directly show that f is uniformly continuous on any interval of the form (a, ∞), provided $a > -1$.

   (b) On the other hand, prove that f is not uniformly continuous on $(-1, \infty)$ by showing that for each $0 < \delta < \frac{1}{2}$, there exists $x, y \in (-1, \infty)$ such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq 1$.

4. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be *Lipschitz continuous* on I if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \qquad \text{for all } x, y \in I.$$

   (a) Prove that if f is Lipschitz continuous on I, it is uniformly continuous on I.

   (b) Suppose that f is differentiable on I and that $f'(x)$ is a bounded function on I. Prove that f is Lipschitz continuous (and hence uniformly continuous).

5. Wade, Exercise 3.4.6.

   Hint: One way to proceed is to use sequences: start by arguing that if f were unbounded, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in I such that $|f(x_n)| \to \infty$. Then apply Bolzano-Weierstrass to the x_n's to extract a subsequence which is Cauchy.

On your own: Wade, 1.3.0, 1.3.8[1], 3.4.1(a,b), 3.4.2(a,b,c), and the following problems:

1. Suppose $a, b, c, d \in \mathbb{R}$ and that $a \leq b \leq c \leq d$. Show that $c - b \leq d - a$. (This isn't hard, but the principle comes up frequently, so gaining familiarity with it is worthwhile.)

[1] Essentially the exercise is asking you to show $\sup(A \cup B) = \max(\sup A, \sup B)$.

2. Suppose $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ is a bounded function. Given a nonempty set $E \subset X$, use Problem 2 above to show that

$$\sup_{t \in E} f(t) - \inf_{t \in E} f(t) = \sup\{|f(t) - f(s)| : t, s \in E\} = \sup\{f(t) - f(s) : t, s \in E\}.$$

   Note: This is nearly immediate from Problem 2 above (just set $A = f(E)$!), but is important to acknowledge since it is used in the theory of the Riemann integral.

3. (a) Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some $k > 0$, then f is uniformly continuous on $[0, \infty)$.

   (b) Show that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.
   Hint: Don't forget about #4b above.

Reading: Wade, Sections 1.3, 3.4. Start Chapter 5 for week 2 of class. Review limits at infinity (e.g. §3.2 in the Wade text).