Math 401/501, Fall 2018
Assignment 9, due Wednesday, October 31

Exercises to hand in:

1. Ross, Exercise 15.6.
2. Ross, Exercise 17.5.
3. Ross, Exercise 17.9.
4. Ross, Exercise 17.10.
5. Ross, Exercise 18.2.
6. A *rational function* is a function of the form $f(x) = p(x)/q(x)$ defined for $x \in \mathbb{R}$ where $p(x) = \sum_{j=0}^{k} \alpha_j x^j$, $q(x) = \sum_{l=0}^{m} \beta_l x^l$ are both polynomials. Suppose $a_n = p(n)/q(n)$ where $p(n) \neq 0$ and $q(n) \neq 0$ for every n. Prove that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{p(n+1)}{p(n)} \cdot \frac{q(n)}{q(n+1)} \right| = 1.$$

   Using Corollary 12.3, conclude that the root and ratio tests *always* fail to provide information when a_n is a rational function.

   Notes: For the sake of discussion, take $p(x), q(x)$ to be polynomials of degree k and m respectively in your solution so that $p(x) = \sum_{j=0}^{k} \alpha_j x^j$, $q(x) = \sum_{l=0}^{m} \beta_l x^l$ with $\alpha_k \neq 0$, $\beta_m \neq 0$. This result shows that there is a broad class of series for which the root and ratio tests fail to provide information, e.g. $\sum \frac{n^3}{2n^5-1}$, $\sum \frac{n+1}{n^2-2}$, etc. So in these cases, you must use the comparison test or another test to deduce convergence or divergence of the series.

On your own: Ross, Exercises 15.1, 15.3, 17.3, 17.4, 18.1, 18.4[1].

1. Suppose $(s_n)_{n=1}^{\infty}$ is a sequence of positive numbers converging to a positive number $s > 0$. Prove that $\lim_{n\to\infty} s_n^{1/n} = 1$.

   Hint: Use Theorem 9.7(d) and the squeeze lemma.
2. Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if and only if $p > 0$.
3. Let $\sum_{k=1}^{\infty} a_k$ be a series and let $s_n = \sum_{k=1}^{n} a_k$ denote the n-th partial sum of the series.
   (a) Prove that if $\sum_{k=1}^{\infty} a_k$ is convergent, then $(s_n)_{n=1}^{\infty}$ is a bounded sequence.
   (b) Suppose that the a_k are nonnegative, i.e. $a_k \geq 0$ for all k. Show that if $(s_n)_{n=1}^{\infty}$ is a bounded sequence, then $\sum_{k=1}^{\infty} a_k$ converges.
   (c) Give a counterexample which shows that the implication in part (b) is generally false if the a_k are not assumed to be nonnegative.

Reading: Ross, §17, 18, 19.

[1] Hint: Try $f(x) = \frac{1}{x-x_0}$.