Math 401/501, Fall 2018
Assignment 4, due Wednesday, September 19

Exercises to hand in:

1. Ross, Exercise 1.6.
2. Ross, Exercise 1.8. Hint for 8b: First prove that if $n \geq 4$, then $n+1 = (1+\frac{1}{n})n \leq 2n < n^2$.
3. Ross, Exercise 3.6.
4. Let $x, y \in \mathbb{R}$ be such that for all $\epsilon > 0$, $x \leq y + \epsilon$. Show that $x \leq y$. Is this different from Exercise 3.8 in Ross?

   Hint: A proof by contrapositive is effective here.
5. Ross, Exercise 4.16.
6. (a) Suppose $x \in \mathbb{Q}$ with $x \neq 0$ and $y \in \mathbb{R} - \mathbb{Q}$. Show that their product xy is irrational.

   (b) Prove the density of $\mathbb{R} - \mathbb{Q}$: Show that given $a, b \in \mathbb{R}$ and $a < b$, then there exists $s \in \mathbb{R} - \mathbb{Q}$ such that $a < s < b$.
   Hint: Apply the density of rationals to $a/\sqrt{2}$ and $b/\sqrt{2}$.
7. Let $S \subset \mathbb{R}$ be a nonempty bounded subset. Given $k \in \mathbb{R}$ define $kS = \{ks : s \in S\}$. Prove the following:

   (a) If $k \geq 0$, then $\sup(kS) = k \cdot \sup S$.

   (b) If $k < 0$, then $\sup(kS) = k \cdot \inf S$.

On your own:

- Ross, Exercises 1.1, 1.3, 1.5, 1.9, 1.11, 2.1, 2.3, 3.5, 3.7, 4.1, 4.3, 4.7, 4.15, 5.1.
- Complement Exercise 7 above by showing that in addition to the identities there, $\inf(kS) = k \cdot \inf S$ if $k \geq 0$ and $\inf(kS) = k \cdot \sup S$ if $k < 0$. Achieving this by appealing directly to the definitions will be more insightful, while appealing to the reflection principle $-\sup(-S) = \inf S$ in Corollary 4.5 will be easier.
- Complement Exercise 6a by showing that
  1. The product of two rational numbers is rational.
  2. There exist two irrational numbers whose product is rational.
  3. If $x, y \in \mathbb{R}$ and $xy \in \mathbb{Q} - \{0\}$, then either both x, y are rational or they are both irrational. (This actually follows easily from Exercise 6a.)

Reading: Ross Chapter 1.