Math 401/501, Fall 2018
Assignment 13, due Wednesday, December 5

Exercises to hand in:

1. Ross, Exercise 32.6.
2. Define $f : [0, 1] \to \mathbb{R}$ as $f(x) = x$ for rational x and $f(x) = 0$ for irrational x. Show that for each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[0, 1]$, we have that $U(f, P) > \frac{1}{2}$. Use this to conclude that f is not integrable on $[0, 1]$.

   Hint: Show that $M(f, [t_{k-1}, t_k]) = t_k$ and that $t_k > \frac{1}{2}(t_k + t_{k-1})$.
3. (a) Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Prove that if $S \subset [a, b]$,

   $$M(f, S) - m(f, S) = \sup\{|f(x) - f(y)| : x, y \in S\}$$

   Hint: This is a variation on an exercise from Assignment 5. Consider different strategies to see that $M(f, S) - m(f, S) \le \sup\{|f(x) - f(y)| : x, y \in S\}$ and $M(f, S) - m(f, S) \ge \sup\{|f(x) - f(y)| : x, y \in S\}$. For the former, try "approximating the supremum/infimum": given $\epsilon > 0$ consider $x, y \in S$ such that $M(f, S) - \epsilon/2 < f(x)$, $m(f, S) + \epsilon/2 > f(y)$, but justify why these exist to begin with!

   (b) Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded functions and that there exists a uniform constant $C > 0$ such that $|f(x) - f(y)| \le C|g(x) - g(y)|$ for all $x, y \in [a, b]$. Show that if g is integrable, then f is also integrable.
4. (a) Use Exercise 3b above to show that if g is integrable on $[a, b]$, then so is g^2.

   (b) Show that if f, g are integrable functions on $[a, b]$, then so is their product fg.
   Hint: Start by showing that $4fg = (f + g)^2 - (f - g)^2$.

On your own: Ross, Exercises 32.7[1], 33.13. Also the following:

1. Suppose $a, b, c, d \in \mathbb{R}$ and that $a \le b \le c \le d$. Show that $c - b \le d - a$, with equality if and only if $a = b = c = d$.

   Note: This is not hard, but this principle comes up frequently in these exercises and beyond, so it is worthwhile to be very familiar with it.
2. Suppose $g : [a, b] \to \mathbb{R}$ is integrable and that there is a uniform constant $c > 0$ such that $g(x) \ge c$ for all $x \in [a, b]$. Use Exercise #3b above to show that the following functions are also integrable on $[a, b]$

   (a) $\sqrt{g}$. Hint: Begin by showing that $\sqrt{g(x)} - \sqrt{g(y)} = \frac{g(x)-g(y)}{\sqrt{g(x)}+\sqrt{g(y)}}$.

   (b) $\frac{1}{g}$.
3. Let $E = \{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that the function $f : [0, 1] \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 1 & x \in E, \\ 0 & x \notin E \end{cases}$$

   is integrable on $[0, 1]$. What is the value of $\int_0^1 f(x)\,dx$?

Reading: Ross, §32, 33, 34.

[1] This is an important exercise even though it is not collected. Justify all steps in the hint in the back.