Math 401/501, Fall 2018
Assignment 12, due Wednesday, November 28

Exercises to hand in:

1. Ross, Exercise 29.15

2. Use Taylor's theorem, prove that if $x \in [0, 1]$, then
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \le \ln(1+x) \le x - \frac{x^2}{2} + \frac{x^3}{3}.$$
You may use all usual properties of the natural logarithm from calculus, including that $\frac{d}{dy} \ln y = \frac{1}{y}$. The choice of remainder $R_n(x)$ will differ when proving each inequality.

3. (a) Suppose that $g : (a, b) \to \mathbb{R}$ is continuous at a point $x_0 \in (a, b)$ with $g(x_0) > 0$. Prove there exists $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta)$ implies $g(x) > \frac{1}{2} g(x_0) > 0$.

   (b) Suppose that $f : (a, b) \to \mathbb{R}$ has continuous derivatives of up to order n, i.e. each derivative $f^{(k)}$ exists and is continuous on (a, b) for $k = 1, \ldots, n$. Assume further that at some $x_0 \in (a, b)$, the first $n - 1$ derivatives of f all vanish at x_0, that is, $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$. Prove that if $f^{(n)}(x_0) > 0$ and n is even, then f has a strict *local minimum at* x_0, i.e. there exists $\delta > 0$ such that
$$f(x) > f(x_0) \text{ for each } x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}.$$

4. Use Taylor's theorem to prove the following version of L'Hôpital's rule: Suppose that f, g are functions on (a, b) with continuous derivatives of up to order n and that
$$f(x_0) = f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0,$$
$$g(x_0) = g'(x_0) = \cdots = g^{(n-1)}(x_0) = 0.$$
Prove that if $g^{(n)}(x_0) \neq 0$, then
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f^{(n)}(x_0)}{g^{(n)}(x_0)}.$$
Note: The point of the problem is to show that under the given hypotheses, L'Hôpital's rule follows from Taylor's theorem. Do not appeal to ideas or theorems in §30.

On your own: Ross, Exercise 29.16.

1. Show that if the hypothesis "n is even" in Exercise 3b replaced by "n is odd", then f has neither a local maximum or local minimum at x_0.

2. Let $f(x) = \cos x$ and let $R_n(x)$ be as in the statement of Taylor's theorem. Prove that if $x \in [-1, 1]$, then
$$\left| \cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} \right| \le \frac{1}{(2n+1)!}.$$
Can the right hand side be strengthened to $\frac{1}{(2n+2)!}$? You may of course assume all differentiability properties of $\cos x$ from calculus.

Reading: Ross, §31, 32, 33. Note: Expect to finish out the course covering §32, 33, 34.