Math 311–Practice for Hour Exam 3

(Note: treat these questions as good practice but not necessarily as indicative of the length of the exam)

1. Suppose $\mathbf{F}(x, y, z)$ a continuously differentiable vector field. You are not given an explicit expression for $\mathbf{F}$, but you do know that over S_1 and S_2, the spheres of radius 1 and 2 about the origin,

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = 6, \qquad S_1 : x^2 + y^2 + z^2 = 1,$$
$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = 2, \qquad S_2 : x^2 + y^2 + z^2 = 4,$$

where in both cases $\mathbf{n}$ is a unit normal which points outward, away from the origin.

Let D be the region consisting of all points (x, y, z) such that $1 \leq x^2 + y^2 + z^2 \leq 4$.
Use the divergence theorem to find

$$\iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Solution: The divergence theorem tells us that the triple integral is equal to the flux of $\mathbf{F}$ through the boundary surface with normal vectors oriented away from the region D. However, the information given regarding the flux of $\mathbf{F}$ through S_1 and S_2 is given with respect to normals pointing away from the origin. Therefore, we must flip the sign on the S_2 integral, in particular

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS - \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = 6 - 2 = 4.$$

2. Let C be the curve formed as the intersection of the plane $y + z = 4$ and the cylinder $x^2 + y^2 = 4$, oriented counterclockwise when viewed from above. Use Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{R}$ where

$$\mathbf{F} = 4x\mathbf{j} + (x + y^3)\mathbf{k}.$$

(Note: your solution *must* involve Stokes' Theorem for full credit.)

Solution: It can be computed that $\nabla \times \mathbf{F} = 3y^2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Furthermore, it can be seen that C encloses the portion of the plane $y + z = 4$ which lies inside the cylinder $x^2 + y^2 = 4$. The latter surface is easily parameterized by

$$\mathbf{R}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y)\mathbf{k}, \qquad x^2 + y^2 \leq 4, \qquad \text{with} \qquad \frac{\partial \mathbf{R}}{\partial x} \times \frac{\partial \mathbf{R}}{\partial y} = \mathbf{j} + \mathbf{k}.$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{x^2+y^2 \leq 4} 3\, dxdy = 3(4\pi) = 12\pi.$$

It is worth mentioning that the answer can be checked by computing the line integral directly.

3. Let S be the the portion of the cylinder $x^2 + y^2 = 1$ with $0 \le z \le 3$, oriented with normals pointing outward (away from the origin). Use the divergence theorem to find

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \qquad \text{where } \mathbf{F} = e^{y^2}\mathbf{i} + 2yz\mathbf{j} - z^2\mathbf{k}.$$

(Note that S is not a closed surface, i.e. it is *not* the boundary of a three dimensional domain. You must add in two additional surfaces in order for it to be closed.)

Solution: A simple computation shows that $\nabla \cdot \mathbf{F} = 0$. S itself is not closed, however, if we include the "lids" of the can, we may apply the divergence theorem. To this end, we define S_{top}, S_{bottom} to be the disks $x^2 + y^2 \le 1$ lying in the $z = 3$, $z = 0$ plane respectively. Thus the surface formed as the union of S, S_{top}, S_{bottom} bounds the solid V determined by the collection of points satisfying $x^2 + y^2 \le 1$, $0 \le z \le 3$. That is,

$$0 = \iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_{S_{top}} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_{bottom}} \mathbf{F} \cdot d\mathbf{S}.$$

Here we must orient S_{top}, S_{bottom} with the unit normals $\mathbf{k}$ and $-\mathbf{k}$ respectively. Thus $\mathbf{F} \cdot \mathbf{n} = -z^2$ on the top and $\mathbf{F} \cdot \mathbf{n} = z^2$ on the bottom. We now obtain

$$\iint_{S_{top}} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_{bottom}} \mathbf{F} \cdot d\mathbf{S} = -9 \cdot \mathbf{Area}(S_{top}) + 0 \cdot \mathbf{Area}(S_{bottom}) = -9\pi.$$

Substituting this back in to the first equation we obtain

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 9\pi.$$

Here again the answer can be checked by a direct computation, but this time it is a little more laborious.

4. a. Suppose $\phi(x, y, z)$ is a harmonic function, i.e. a solution to Laplace's equation $\nabla^2\phi = 0$ (where $\nabla^2\phi = \nabla \cdot (\nabla\phi)$ is the Laplacian of ϕ). Use vector identities to show that

$$\nabla \cdot (\phi\nabla\phi) = |\nabla\phi|^2.$$

Solution: Using vector identities we have that

$$\nabla \cdot (\phi\nabla\phi) = \nabla\phi \cdot \nabla\phi + \phi\nabla \cdot (\nabla\phi) = |\nabla\phi|^2 + 0 = |\nabla\phi|^2 .$$

b. Suppose $\phi(x, y, z)$ is a harmonic function which satisfies $\phi(x, y, z) = 0$ whenever (x, y, z) lies on the sphere $x^2 + y^2 + z^2 = 1$. Let D denote the domain enclosed by the sphere, i.e. the set of points (x, y, z) satisfying $x^2 + y^2 + z^2 \le 1$. Use the divergence theorem and the identity from part **a** to show that

$$\iiint_D |\nabla\phi|^2\, dV = 0.$$

Solution: Applying the identity in part **a** and integrating over D, we obtain

$$\iiint_D |\nabla\phi|^2\, dV = \iiint_D \nabla \cdot (\phi\nabla\phi)\, dV = \iint_S \phi\nabla\phi \cdot \mathbf{n}\, dS = 0$$

with the last equality following from the fact that $\phi(x, y, z) = 0$ for all points on the sphere.