Math 311, Section 2–Solutions to the second practice exam

1. Compute the curl of the vector field $F(r, \phi, \theta) = r e_\phi + e_\theta$ in spherical coordinates.

   Solution: Using the formula for the curl in spherical coordinates with $F_r = 0$, $F_\phi = r$, $F_\theta = 1$ gives
   
   $$\text{curl } F = \frac{1}{r^2 \sin \phi} \begin{vmatrix} e_r & r e_\phi & r \sin \phi e_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ 0 & r^2 & r \sin \phi \end{vmatrix}$$
   
   $$= \frac{1}{r^2 \sin \phi} \left( r \cos \phi e_r - (\sin \phi) r e_\phi + 2r \cdot r \sin \phi e_\theta \right)$$
   
   $$= \frac{1}{r} \cot \phi e_r - \frac{1}{r} e_\phi + 2e_\theta.$$ 

2. Let $F$ be the vector field $F(x, y, z) = z^2 i - \sin y j + (2xz + 6z^2) k$.

   a. Determine whether or not $F$ is conservative. If it is conservative, find a potential function for $F$.

   Solution: Computing $\text{curl } F$ gives $0i - (2z - 2z)j + 0k$ which after simplification is the zero vector. Therefore since the $F$ is continuously differentiable at all points, the zero curl test shows that $F$ is conservative.

   To find a potential function $\phi$, observe that it must satisfy
   
   $$\phi(x, y, z) = \int \frac{\partial \phi}{\partial z} \, dz = \int 2xz + 6z^2 \, dz = xz^2 + 2z^3 + \psi(x, y)$$
   
   where $\psi(x, y)$ is a function of $x$ and $y$ alone. To solve for $\psi$, observe that by what we just computed, the partial derivative of $\phi$ in $x$ must equal
   
   $$\frac{\partial \phi}{\partial x} = z^2 + \frac{\partial \psi}{\partial x}.$$ 

   Setting this equal to $z^2$, the first component of $F$, we see that $\frac{\partial \psi}{\partial x} = 0$, which implies that $\psi$ must be a function of $y$ alone, that is $\psi = \psi(y)$. We now know that
   
   $$\phi(x, y, z) = xz^2 + 2z^3 + \psi(y)$$

   Differentiating this in $y$ gives the expression
   
   $$\frac{\partial \phi}{\partial y} = \psi'(y)$$

   which must equal $-\sin y$. Therefore up to a constant, we have $\psi(y) = \cos y$ and
   
   $$\phi(x, y, z) = xz^2 + 2z^3 + \cos y.$$
b. Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{R} \), where \( C \) is the path parameterized by

\[
\mathbf{R}(t) = te^{1-t} \mathbf{i} + \frac{\pi}{2} t \mathbf{j} + t^4 \mathbf{k}, \quad 0 \leq t \leq 1.
\]

Simplify your answer.

(Hint: Try to use your answer from part a. What are the initial and terminal points of the path?)

Solution: The initial and terminal points of the path can be found by substituting \( t = 0 \) and \( t = 1 \) into the vector equation for the path respectively

\[
\mathbf{R}(0) = (0, 0, 0), \quad \mathbf{R}(1) = (1, \frac{\pi}{2}, 1).
\]

Using the potential function found in part a and the theorem for line integrals we have

\[
\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, \frac{\pi}{2}, 1) - \phi(0, 0, 0) = (1 + 2 + \cos(\pi/2)) - (0 + 0 + \cos(0)) = 3 - 1 = 2
\]

3. Let \( \mathbf{F} \) be the two dimensional vector field \( \mathbf{F} = \sin x \mathbf{i} - y \cos x \mathbf{j} \).

a. Show that \( \mathbf{F} \) is solenoidal.

Solution: Solenoidal fields have zero divergence, that is, \( \nabla \cdot \mathbf{F} = 0 \). A computation of the divergence of \( \mathbf{F} \) yields

\[
\text{div } \mathbf{F} = \cos x - \cos x = 0.
\]

Hence \( \mathbf{F} \) is solenoidal.

b. Find a vector potential for \( \mathbf{F} \).

Solution: The vector field is 2 dimensional, therefore we may use the techniques on p. 221 of the text to find a vector potential. We thus look for a vector potential of the form \( \mathbf{G} = \chi \mathbf{k} \) where \( \chi(x, y) \) is a scalar function of two variables satisfying

\[
F_1 = \frac{\partial \chi}{\partial y}, \quad -F_2 = \frac{\partial \chi}{\partial x}.
\]

The first equation tells us that \( \sin x = \frac{\partial \chi}{\partial y} \) which can be integrated in \( y \) to give that \( \chi \) must be of the form

\[
\chi = y \sin x + \psi(x)
\]

where \( \psi(x) \) is a function of \( x \) alone. Differentiating this expression for \( \chi \) in \( x \) gives \( \frac{\partial \chi}{\partial x} = y \cos x + \psi'(x) \). Since \( -F_2 = y \cos x \), we now know that \( \psi'(x) = 0 \). Therefore we can take \( \chi(x, y) = y \sin x \), and a vector potential for \( \mathbf{F} \) will be

\[
\mathbf{G} = y \sin x \mathbf{k}.
\]

Indeed, it can be checked that \( \text{curl } \mathbf{G} = \mathbf{F} \).
4. Let \( S \) be the portion of the cone \( z^2 = x^2 + y^2 \) lying in the first octant with \( 0 \leq z \leq 2 \) and oriented upward (so that \( \mathbf{n} \cdot \mathbf{k} \) is positive). Evaluate \( \iint_S \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F}(x, y, z) = -yi + xj + k \).

\[ \text{Solution:} \quad \text{Fixing} \quad z = \rho \quad \text{in cylindrical coordinates we may find a parametrization of the cone taking the form} \]
\[ \mathbf{R}(\rho, \theta) = \rho \cos \theta \mathbf{i} + \rho \sin \theta \mathbf{j} + \rho \mathbf{k}. \]

Since \( z = \rho \), we know \( 0 \leq \rho \leq 2 \), as that is the condition on \( z \). Since \( S \) lies in the first octant, every point on the surface will form an angle between 0 and \( \pi/2 \) with the positive \( x \)-axis. Therefore, we have that \( 0 \leq \theta \leq \frac{\pi}{2} \). We now compute the normal vectors
\[ \frac{\partial \mathbf{R}}{\partial \rho} \times \frac{\partial \mathbf{R}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -\rho \sin \theta & \rho \cos \theta & 0 \end{vmatrix} = -\rho \cos \theta \mathbf{i} - \rho \sin \theta \mathbf{j} + \rho \mathbf{k}, \]
which points upward since \( \rho > 0 \). Along the surface parametrization, the vector field takes the form
\[ \mathbf{F}(\mathbf{R}(\rho, \theta)) = -\rho \sin \theta \mathbf{i} + \rho \cos \theta \mathbf{j} + \mathbf{k}. \]

Hence
\[ \mathbf{F}(\mathbf{R}(\rho, \theta)) \cdot \frac{\partial \mathbf{R}}{\partial \rho} \times \frac{\partial \mathbf{R}}{\partial \theta} = \rho^2 \cos \theta \sin \theta - \rho^2 \cos \theta \sin \theta + \rho = \rho. \]

Therefore,
\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^\pi \int_0^2 \rho \ d\rho \ d\theta = \pi \]

5. Let \( V \) be the domain bounded by the cylinder \( x^2 + y^2 = 4 \) which lies below the plane \( z = 2 + x \) and above the \( z = 0 \) plane. Find the integral of \( f(x, y, z) = x^2 + y^2 \) over \( V \), that is, find
\[ \iiint_V x^2 + y^2 \ dV. \]

\[ \text{Show your work.} \]

\[ \text{Solution:} \quad \text{Since} \quad V \quad \text{is bounded by a cylinder and 2 planes, and the function} \quad f \quad \text{is easily expressed in cylindrical coordinates, these coordinates are an ideal choice for this problem. We first observe that in rectangular coordinates} \quad V \quad \text{can be described by the condition that} \quad x^2 + y^2 \leq 4 \quad \text{and} \quad 0 \leq z \leq 2 + x. \quad \text{In cylindrical coordinates, this means that} \quad 0 \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad \text{and} \quad 0 \leq z \leq 2 + \rho \cos \theta. \quad \text{Since} \quad x^2 + y^2 = \rho^2 \quad \text{in cylindrical coordinates, this means that we may compute the triple integral as} \]
\[ \int_0^2 \int_0^{2\rho \cos \theta} \rho^2 \cdot \rho \ dz \ d\theta \ d\rho = \int_0^2 \int_0^{2\pi} \rho^3(2 + \rho \cos \theta) \ d\theta \ d\rho = \int_0^2 \rho^3 [2\theta + \rho \sin \theta]_{\theta=0}^{\theta=2\pi} \ d\rho \]
\[ = 4\pi \int_0^2 \rho^3 \ d\rho = 16\pi. \]