Math 311, Section 2–Solutions to the second practice exam

1. Compute the curl of the vector field $\mathbf{F}(r,\phi,\theta) = r\,\mathbf{e}_\phi + \mathbf{e}_\theta$ in spherical coordinates.

Solution: Using the formula for the curl in spherical coordinates with $F_r = 0$, $F_\phi = r$, $F_\theta = 1$ gives

$$\begin{aligned}
\textbf{curl F} &= \frac{1}{r^2\sin\phi}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\phi & r\sin\phi\,\mathbf{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ 0 & r^2 & r\sin\phi \end{vmatrix} \\
&= \frac{1}{r^2\sin\phi}\left(r\cos\phi\,\mathbf{e}_r - (\sin\phi)r\,\mathbf{e}_\phi + 2r\cdot r\sin\phi\,\mathbf{e}_\theta\right) \\
&= \frac{1}{r}\cot\phi\,\mathbf{e}_r - \frac{1}{r}\,\mathbf{e}_\phi + 2\mathbf{e}_\theta.
\end{aligned}$$

2. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y,z) = z^2\mathbf{i} - \sin y\mathbf{j} + (2xz + 6z^2)\mathbf{k}$.

a. Determine whether or not $\mathbf{F}$ is conservative. If it is conservative, find a potential function for $\mathbf{F}$.

Solution: Computing **curl F** gives $0\mathbf{i} - (2z - 2z)\mathbf{j} + 0\mathbf{k}$ which after simplification is the zero vector. Therefore since the $\mathbf{F}$ is continuously differentiable at all points, the zero curl test shows that $\mathbf{F}$ is conservative.

To find a potential function ϕ, observe that it must satisfy

$$\phi(x,y,z) = \int \frac{\partial\phi}{\partial z}\,dz = \int 2xz + 6z^2\,dz = xz^2 + 2z^3 + \psi(x,y)$$

where $\psi(x,y)$ is a function of x and y alone. To solve for ψ, observe that by what we just computed, the partial derivative of ϕ in x must equal

$$\frac{\partial\phi}{\partial x} = z^2 + \frac{\partial\psi}{\partial x}.$$

Setting this equal to z^2, the first component of $\mathbf{F}$, we see that $\frac{\partial\psi}{\partial x} = 0$, which implies that ψ must be a function of y alone, that is $\psi = \psi(y)$. We now know that

$$\phi(x,y,z) = xz^2 + 2z^3 + \psi(y)$$

Differentiating this in y gives the expression

$$\frac{\partial\phi}{\partial y} = \psi'(y)$$

which must equal $-\sin y$. Therefore up to a constant, we have $\psi(y) = \cos y$ and

$$\phi(x,y,z) = xz^2 + 2z^3 + \cos y.$$

b. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is the path parameterized by

$$\mathbf{R}(t) = te^{1-t}\,\mathbf{i} + \frac{\pi}{2}\,t\,\mathbf{j} + t^4\,\mathbf{k}, \qquad 0 \le t \le 1.$$

Simplify your answer.
(Hint: Try to use your answer from part **a**. What are the initial and terminal points of the path?)

Solution: The initial and terminal points of the path can be found by substituting $t = 0$ and $t = 1$ into the vector equation for the path respectively

$$\mathbf{R}(0) = \langle 0, 0, 0\rangle, \qquad \mathbf{R}(1) = \langle 1, \frac{\pi}{2}, 1\rangle.$$

Using the potential function found in part **a** and the theorem for line integrals we have

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, \frac{\pi}{2}, 1) - \phi(0, 0, 0) = (1 + 2 + \cos(\pi/2)) - (0 + 0 + \cos(0)) = 3 - 1 = 2$$

3. Let $\mathbf{F}$ be the two dimensional vector field $\mathbf{F} = \sin x\,\mathbf{i} - y\cos x\,\mathbf{j}$.

a. Show that $\mathbf{F}$ is solenoidal.

Solution: Solenoidal fields have zero divergence, that is, $\nabla \cdot \mathbf{F} = 0$. A computation of the divergence of $\mathbf{F}$ yields

$$\textbf{div } \mathbf{F} = \cos x - \cos x = 0.$$

Hence $\mathbf{F}$ is solenoidal.

b. Find a vector potential for $\mathbf{F}$.

Solution: The vector field is 2 dimensional, therefore we may use the techniques on p. 221 of the text to find a vector potential. We thus look for a vector potential of the form $\mathbf{G} = \chi\mathbf{k}$ where $\chi(x, y)$ is a scalar function of two variables satisfying

$$F_1 = \frac{\partial \chi}{\partial y}, \qquad -F_2 = \frac{\partial \chi}{\partial x}.$$

The first equation tells us that $\sin x = \frac{\partial \chi}{\partial y}$ which can be integrated in y to give that χ must be of the form

$$\chi = y \sin x + \psi(x)$$

where $\psi(x)$ is a function of x alone. Differentiating this expression for χ in x gives $\frac{\partial \chi}{\partial x} = y\cos x + \psi'(x)$. Since $-F_2 = y\cos x$, we now know that $\psi'(x) = 0$. Therefore we can take $\chi(x, y) = y\sin x$, and a vector potential for $\mathbf{F}$ will be

$$\mathbf{G} = y \sin x\,\mathbf{k}.$$

Indeed, it can be checked that $\textbf{curl } \mathbf{G} = \mathbf{F}$.

4. Let S be the portion of the cone $z^2 = x^2 + y^2$ lying in the first octant with $0 \le z \le 2$ and oriented upward (so that $\mathbf{n} \cdot \mathbf{k}$ is positive). Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$.

Solution: Fixing $z = \rho$ in cylindrical coordinates we may find a parametrization of the cone taking the form

$$\mathbf{R}(\rho, \theta) = \rho \cos\theta \mathbf{i} + \rho \sin\theta \mathbf{j} + \rho \mathbf{k}.$$

Since $z = \rho$, we know $0 \le \rho \le 2$, as that is the condition on z. Since S lies in the first octant, every point on the surface will form an angle between 0 and $\pi/2$ with the positive x-axis. Therefore, we have that $0 \le \theta \le \frac{\pi}{2}$. We now compute the normal vectors

$$\frac{\partial \mathbf{R}}{\partial \rho} \times \frac{\partial \mathbf{R}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} = -\rho\cos\theta\,\mathbf{i} - \rho\sin\theta\,\mathbf{j} + \rho\,\mathbf{k},$$

which points upward since $\rho > 0$. Along the surface parametrization, the vector field takes the form $\mathbf{F}(\mathbf{R}(\rho, \theta)) = -\rho\sin\theta\mathbf{i} + \rho\cos\theta\mathbf{j} + \mathbf{k}$. Hence

$$\mathbf{F}(\mathbf{R}(\rho, \theta)) \cdot \frac{\partial \mathbf{R}}{\partial \rho} \times \frac{\partial \mathbf{R}}{\partial \theta} = \rho^2 \cos\theta \sin\theta - \rho^2 \cos\theta \sin\theta + \rho = \rho$$

Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}} \int_0^2 \rho \; d\rho \, d\theta = \pi$$

5. Let V be the domain bounded by the cylinder $x^2 + y^2 = 4$ which lies below the plane $z = 2 + x$ and above the $z = 0$ plane. Find the integral of $f(x, y, z) = x^2 + y^2$ over V, that is, find

$$\iiint_V x^2 + y^2 \; dV.$$

Show your work.

Solution: Since V is bounded by a cylinder and 2 planes, and the function f is easily expressed in cylindrical coordinates, these coordinates are an ideal choice for this problem. We first observe that in rectangular coordinates V can be described by the condition that $x^2 + y^2 \le 4$ and $0 \le z \le 2 + x$. In cylindrical coordinates, this means that $0 \le \rho \le 2$, $0 \le \theta \le 2\pi$, and $0 \le z \le 2 + \rho\cos\theta$. Since $x^2 + y^2 = \rho^2$ in cylindrical coordinates, this means that we may compute the triple integral as

$$\begin{aligned} \int_0^2 \int_0^{2\pi} \int_0^{2+\rho\cos\theta} \rho^2 \cdot \rho \; dz \, d\theta \, d\rho &= \int_0^2 \int_0^{2\pi} \rho^3 (2 + \rho\cos\theta) \; d\theta \, d\rho = \int_0^2 \rho^3 \left[2\theta + \rho\sin\theta\right]_{\theta=0}^{\theta=2\pi} \; d\rho \\ &= 4\pi \int_0^2 \rho^3 \; d\rho = 16\pi \end{aligned}$$