Math 311 Practice Exam 2

1. Compute the curl of the vector field \( \mathbf{F}(r, \phi, \theta) = r \mathbf{e}_{\phi} + \mathbf{e}_{\theta} \) in spherical coordinates.

2. Let \( \mathbf{F} \) be the vector field \( \mathbf{F}(x, y, z) = z^2 \mathbf{i} - \sin y \mathbf{j} + (2xz + 6z^2) \mathbf{k} \).
   
   a. Determine whether or not \( \mathbf{F} \) is conservative. If it is conservative, find a potential function for \( \mathbf{F} \).
   
   b. Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{R} \), where \( C \) is the path parameterized by
      \[
      \mathbf{R}(t) = te^{1-t} \mathbf{i} + \frac{\pi}{2} t \mathbf{j} + t^4 \mathbf{k}, \quad 0 \leq t \leq 1.
      \]
      Simplify your answer. (Hint: Try to use your answer from part a. What are the initial and terminal points of the path?)

3. Let \( \mathbf{F} \) be the two dimensional vector field \( \mathbf{F} = \sin x \mathbf{i} - y \cos x \mathbf{j} \).
   
   a. Show that \( \mathbf{F} \) is solenoidal.
   
   b. Find a vector potential for \( \mathbf{F} \).

4. Let \( S \) be the portion of the cone \( z^2 = x^2 + y^2 \) lying in the first octant with \( 0 \leq z \leq 2 \) and oriented upward (so that \( \mathbf{n} \cdot \mathbf{k} \) is positive). Evaluate \( \oiint_S \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F}(x, y, z) = -y \mathbf{i} + x \mathbf{j} + \mathbf{k} \).

5. Let \( V \) be the domain bounded by the cylinder \( x^2 + y^2 = 4 \) which lies below the plane \( z = 2 + x \) and above the \( z = 0 \) plane. Find the integral of \( f(x, y, z) = x^2 + y^2 \) over \( V \), that is, find
   \[
   \iiint_V x^2 + y^2 \, dV.
   \]
   Show your work.