Math 511
Assignment 8, due Thursday, April 4

1. (Wade 12.4.6) Suppose that V is nonempty and open in $\mathbb{R}^n$ and that $\phi : V \to \mathbb{R}^n$ is continuously differentiable with $\Delta_\phi(x) = \det(\phi'(x)) \neq 0$ for all $x \in V$. Prove that

$$\lim_{r \to 0^+} \frac{\mathrm{Vol}(\phi(B_r(x_0)))}{\mathrm{Vol}(B_r(x_0))} = |\Delta_\phi(x_0)|$$

2. (Wade 12.4.9) Let $\{v_1, \ldots, v_n\}$ be a set of n vectors in $\mathbb{R}^n$. The *parallelepiped* determined by these vectors is the set

$$\mathcal{P}(v_1, \ldots, v_n) := \{t_1 v_1 + t_2 v_2 + \cdots + t_n v_n : t_j \in [0, 1]\}.$$

and that $\det(v_1, \ldots, v_n)$ is the determinant of the matrix whose columns are $v_1, \ldots v_n$. Prove that

$$\mathrm{Vol}(\mathcal{P}(v_1, \ldots, v_n)) = |\det(v_1, \ldots, v_n)|.$$

(This is a change of variables exercise when the v_j are linearly independent. Be rigorous about the linearly dependent case.)

3. A mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an *affine transformation* if it is defined by $\phi(x) = Ax + b$, where A is a non-singular $n \times n$ matrix, Ax denotes matrix vector multiplication, and $b \in \mathbb{R}^n$. Suppose $E \subset \mathbb{R}^n$ is a Jordan region and that ϕ is an affine transformation.

   (a) Show that $\mathrm{Vol}(\phi(E)) = |\det A| \times \mathrm{Vol}(E)$.

   (b) The *centroid* of E is defined as the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ where

$$\bar{x}_i = \frac{1}{\mathrm{Vol}(E)} \int_E x_i \, dV$$

   where the integral on the right is to be interpreted as the integral of the function $g(x) = x_i$ over the region E. Show that $\phi(\bar{x})$ is the centroid of $\phi(E)$.

4. Let B_R denote the ball of radius $R > 0$ about the origin in $\mathbb{R}^2$. Suppose $f : \overline{B}_1 \to \mathbb{R}$ is a continuous function.

   (a) Show that $\lim_{R \to 0^+} \iint_{B_R} f(x, y) \, dx \, dy = 0$.

   (b) Explain why the change of variables theorem for Riemann integrable functions by itself just barely falls short of implying the polar coordinates formula

$$\iint_{B_1} f(x, y) \, dx \, dy = \int_0^1 \int_0^{2\pi} f(r \cos\theta, r \sin\theta) \, r \, d\theta \, dr.$$

(c) Supplement part (a) to prove that this formula is valid anyway.

On your own: Do exercises, 12.4.7, 12.4.10 in Wade and the following exercises in Bachman (2nd edition): 3.5-3.12, 3.14, 3.24. While the exercises in Bachman are relatively short and will not be collected, they are essential. 3.5-3.12 are consequences of properties of the determinant discussed in Chapter 9 of Rudin.

Reading: Bachman, Chapter 3 and 4.1-4.5, though 3.5 is not crucial. Review Chapters 1 and 2 as needed, in particular surface parameterizations.