Math 511
Assignment 5, due Thursday, February 21

Exercises to hand in:

1. Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is C^2 and that $f_y(0, 0) \neq 0$, $f(0, 0) = 0$.

   (a) Show that there exists a neighborhood $(-\epsilon, \epsilon)$ and a continuously differentiable, real valued function ϕ defined on this set such that $\phi(0) = 0$ and $f(x, \phi(x)) = 0$.

   (b) Show that the vector $\langle 1, \phi'(x)\rangle$ is orthogonal to the vector

   $$\langle f_x(x, \phi(x)), f_y(x, \phi(x))\rangle$$

   for all $x \in (-\epsilon, \epsilon)$.

   (c) Now define the map $F(x, w) = (x+wf_x(x, \phi(x)), \phi(x)+wf_y(x, \phi(x)))$. Show that F is one-to-one in a neighborhood of the origin.

2. Suppose $F : \Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^{n+m}$ is an open set containing the origin and that $F \in \mathcal{C}'(\Omega)$. Suppose further that $F(0) = 0$ and $F'(0)$ has full rank, in other words the rank of $F'(0)$ is n. *Without using the rank theorem*, show that if $c \in \mathbb{R}^n$ is sufficiently close to 0, then there exists $(x, y) \in \Omega$ such that $F(x, y) = c$, that is, $F(x, y) = c$ admits a solution. (Hint: the inverse function theorem is effective here.)

3. Suppose $\Omega \subset \mathbb{R}^n$ is a convex open set and $f : \Omega \to \mathbb{R}$ is such that $(D_1 f)(x) = 0$ for every $x \in E$. Prove that $f(x)$ depends only on $x_2, \ldots, x_n$. (This is the first half of #10 Ch. 9, Rudin. Think about the second half on your own, but you do not need to hand it in.)

4. Rudin, Chapter 9, # 24

On your own: Rudin, Chapter 9: #19 and the following problem:
Let $\psi \in C^1(\mathbb{R}^2)$ be a real valued function with nowhere vanishing gradient, $\psi = \psi(u, v)$ and $a, b \in \mathbb{R}$ two constants such that

$$a\frac{\partial \psi}{\partial u} + b\frac{\partial \psi}{\partial v} \neq 0$$

1. Show that the equation $\psi(x + az, y + bz) = 0$ defines z implicitly as a function of (x, y), $z = z(x, y) \in \mathbb{R}$.

2. Show that $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = -1$

Reading in Rudin: Chapter 9