Exercises to hand in:

1. Suppose \( f(x, y) : \mathbb{R}^2 \to \mathbb{R} \) is \( C^2 \) and that \( f_y(0, 0) \neq 0, \) \( f(0, 0) = 0. \)

   (a) Show that there exists a neighborhood \((-\epsilon, \epsilon)\) and a continuously differentiable, real valued function \( \phi \) defined on this set such that \( \phi(0) = 0 \) and \( f(x, \phi(x)) = 0. \)

   (b) Show that the vector \( \langle 1, \phi'(x) \rangle \) is orthogonal to the vector \( \langle f_x(x, \phi(x)), f_y(x, \phi(x)) \rangle \)

   for all \( x \in (-\epsilon, \epsilon). \)

   (c) Now define the map \( F(x, w) = (x + w f_x(x, \phi(x)), \phi(x) + w f_y(x, \phi(x))). \)

   Show that \( F \) is one-to-one in a neighborhood of the origin.

2. Suppose \( F : \Omega \to \mathbb{R}^n \) where \( \Omega \subset \mathbb{R}^{n+m} \) is an open set containing the origin and that \( F \in C^r(\Omega). \) Suppose further that \( F(0) = 0 \) and \( F'(0) \) has full rank, in other words the rank of \( F'(0) \) is \( n. \)

   Without using the rank theorem, show that if \( c \in \mathbb{R}^n \) is sufficiently close to 0, then there exists \( (x, y) \in \Omega \) such that \( F(x, y) = c, \) that is, \( F(x, y) = c \) admits a solution.

   (Hint: the inverse function theorem is effective here.)

3. Suppose \( \Omega \subset \mathbb{R}^n \) is a convex open set and \( f : \Omega \to \mathbb{R} \) is such that \( (D_1 f)(x) = 0 \) for every \( x \in E. \) Prove that \( f(x) \) depends only on \( x_2, \ldots, x_n. \)

   (This is the first half of \#10 Ch. 9, Rudin. Think about the second half on your own, but you do not need to hand it in.)

4. Rudin, Chapter 9, \# 24

   On your own: Rudin, Chapter 9: \#19 and the following problem:

   Let \( \psi \in C^1(\mathbb{R}^2) \) be a real valued function with nowhere vanishing gradient, \( \psi = \psi(u, v) \) and \( a, b \in \mathbb{R} \) two constants such that

   \[
   a \frac{\partial \psi}{\partial u} + b \frac{\partial \psi}{\partial v} \neq 0
   \]

   1. Show that the equation \( \psi(x + az, y + bz) = 0 \) defines \( z \) implicitly as a function of \( (x, y), \) \( z = z(x, y) \in \mathbb{R}. \)

   2. Show that \( a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = -1 \)

   Reading in Rudin: Chapter 9