Math 511
Assignment 4, due Thursday, February 14

Exercises to hand in:

1. Let Ω be an open subset of $\mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^m$ be differentiable on Ω. Let $a, b \in \Omega$ be two points such that the line segment joining a, b lies entirely in Ω. Given any $u \in \mathbb{R}^m$, show that there exists c on this line segment such that

$$u \cdot (F(b) - F(a)) = u \cdot (F'(c)(b - a)).$$

   Then use this to give a second proof of Theorem 9.19 in Rudin.

2. Let Ω be an open subset of $\mathbb{R}^n$ such that every two points $x, y \in \Omega$ can be joined by a continuously differentiable curve of length less than or equal to $100|x - y|$, that is, there exists $\gamma_{x,y} : [0, 1] \to \Omega$ such that

$$\int_0^1 \left|\gamma'_{x,y}(t)\right| \, dt \leq 100|x - y|.$$

   Show that if F is a continuously differentiable map $F : \Omega \to \mathbb{R}^n$, such that for some constant M we have $\|F'(x)\| \leq M$ for all $x \in \Omega$, then F is uniformly Lipschitz, that is, for some constant Λ independent of $x, y \in \Omega$ we have

$$|F(x) - F(y)| \leq \Lambda|x - y|.$$

3. Suppose $\Omega \subset \mathbb{R}^n$ is open and $F : \Omega \to \mathbb{R}^n$ is a continuously differentiable map such that $\det(F'(x)) \neq 0$ for every $x \in \Omega$. Show that the image of Ω under F,

$$F(\Omega) = \{F(x) : x \in \Omega\}$$

   is open.

4. Rudin, Chapter 9, # 16

On your own: Rudin, Chapter 9: #17
Reading in Rudin: Chapter 9