Math 510, Fall 2017
Assignment 4, due Wednesday, September 20

Hand in Parts I and II of this assignment separately:

Part I

1. Let (X, d) be a metric space. Let E be a nonempty subset of X. Define the distance from $p \in X$ to E by

$$\rho_E(p) = \inf \{d(p, q) : q \in E\}.$$

   Prove that $\rho_E(x) = 0$ if and only if x belongs to $\overline{E}$.

2. Rudin, Chapter 2, #12.

3. Rudin, Chapter 2, #16. (Using relatively open and closed sets allows for an efficient answer to some of these questions)

Part II

1. Let (X, d) be a metric space and suppose that F is a closed subset of X and that K is a compact subset of X. The distance between F and K is defined to be

$$d(F, K) := \inf\{d(p, q) : \ p \in F, \ q \in K\}.$$

   (a) Show that if the sets F and K are disjoint then $d(F, K) > 0$.

   (b) Is the result true if K is only assumed to be closed? Justify your answer.

   Hint for (a): For each $p \in K$, there exists $r_p > 0$ such that $N_{r_p}(p) \subset F^C$. Consider the open cover of K given by $\{N_{\frac{r_p}{2}}(p)\}_{p \in K}$.

2. The following problem considers $\mathbb{R}^k$ endowed with the usual metric. Denote $\mathbb{Q}^k = \{y = (y_1, \ldots, y_k) \in \mathbb{R}^k : y_j \in \mathbb{Q} \text{ for each } j\}$.

   (a) Suppose $G \subset \mathbb{R}^k$ is an open subset. Show that for any $x \in G$, there exists $y \in \mathbb{Q}^k$ and a rational radius $r \in \mathbb{Q}$ such that $x \in N_r(y)$ and $N_r(y) \subset G$.

   (b) Suppose $E \subset \mathbb{R}^k$ is any subset and that $\{G_\alpha\}_{\alpha \in A}$ is an open cover of E, that is, $E \subset \cup_{\alpha \in A} G_\alpha$ and G_α is open for each $\alpha \in A$. Prove that there exists a countable subcover of E in that there exists a countable subset $A_0 \subset A$ such that $E \subset \cup_{\alpha \in A_0} G_\alpha$.

   Note: Despite its appearance, this problem has nothing to do with compactness.

   Hint: Use part (a) to help you: begin by showing that there is a countable collection of neighborhoods $\{N_{r_j}(y_j)\}_{j=1}^{\infty}$ which cover E and $N_{r_j}(y_j) \subset V_\alpha$ for some $\alpha \in A$.

On your own (i.e. not collected or graded):

1. Determine which of the following sets in $\mathbb{R}^2$ (with the usual metric) is compact. If any of them is not compact, find the smallest compact set (if it exists) containing the given set.

   (a) $\{(x, y) \in \mathbb{R}^2 : y = \sin(1/x) \text{ for some } x \in (0, 1)\}$

   (b) $\{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$

2. Rudin, Chapter 2, #15

3. Let X be a nonempty set endowed with the discrete metric. Prove that $E \subset X$ is compact if and only if it is finite.

Reading: Finish Chapter 2.