Math 401/501
Assignment 12, due Thursday, December 5 (Last class)

Problems from Taylor:
§5.1, p.107-8: 4, 5
§5.2, p.113-4: 8, 9, 10

Also, hand in the following extra problem:

(a) Suppose $h : [a, b] \to \mathbb{R}$ be a function such that $h(x) = 0$ except at finitely many points. In other words, $S = \{s \in [a, b] : h(s) \neq 0\}$ is a finite set $S = \{s_1, \ldots, s_\ell\}$. Prove that h is integrable on $[a, b]$ and $\int_a^b h(x)\,dx = 0$.

Hint: one way to approach this is to consider a partition $P_n = \{x_0, \ldots, x_n\}$ of equally spaced points so that $x_k - x_{k-1} = (b - a)/n$ for $k = 1, \ldots, n$ and prove that

$$U(f, P_n) - L(f, P_n) \leq \frac{4M\ell(b - a)}{n}$$

where $M = \sup_{[a,b]} |h| = \max\{|h(s_1)|, |h(s_2)|, \ldots, |h(s_\ell)|\}$.

(b) Conclude that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $g : [a, b] \to \mathbb{R}$ is a function such that $\{x \in [a, b] : f(x) \neq g(x)\}$ is finite, then g is also Riemann integrable and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

Notes:

- For #8 in 5.2, it may be helpful to observe that

$$\sup\{|f(x) - f(y)| : x, y \in I\} = \sup\{f(x) - f(y) : x, y \in I\}.$$

  Indeed, if M is any upper bound for the set on the left hand side, then it must also be an upper bound for the set on the right hand side, showing that the left hand side is greater than or equal to the right hand side. On the other hand it can be seen that

$$\{|f(x) - f(y)| : x, y \in I\} \subset \{f(x) - f(y) : x, y \in I\},$$

  showing that the left hand side is less than or equal to the right hand side by Theorem 1.5.7(d).

- Theorem 1.5.10(d) and the identity in the preceding note may also be helpful in 5.2 #9.

Not collected:
§5.1, p.107-8: 2, 3, 9
§5.2, p.113-4: 2, 5

Reading: 5.1, 5.2, 5.3