COMPLEX ANALYSIS QUALIFYING EXAM

JANUARY 8, 2018

Department of Mathematics and Statistics
University of New Mexico

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1. Let $f(z) = \dfrac{z^3}{(z^2+1)e^{1/z}}$.

   (a) Find and classify all the singularities of $f(z)$ in the extended complex plane.

   (b) Does $|f(z)|$ converge to infinity as $|z|$ approaches zero? Justify your answer.

   (c) Find the coefficient of $\frac{1}{z}$ in the Laurent series of $f(z)$ in the domain $\{z \in \mathbb{C} : |z| < 1\}$.

   (d) Find the coefficient of $\frac{1}{z}$ in the Laurent series of $f(z)$ in the domain $\{z \in \mathbb{C} : |z| > 1\}$.

   (e) Evaluate $\oint_\Gamma f(z)dz$, where $\Gamma = \{z \in \mathbb{C} : |\mathrm{Re} z| + |\mathrm{Im} z| = 3\}$.

2. Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$. Justify your calculations.

3. (a) Find the image of the positively oriented strip $\{z = x + iy : y \in [0, \pi], x \in \mathbb{R}\}$ under the map $w = e^z$, including its orientation. Label 5 relevant points A, B, C, D, E of your choice in the z-plane and their images in the w-plane.

   (b) Let $D = \{z \in \mathbb{C} : |z| \le 1 \text{ or } |z-2| \le 1\}$ and $\Omega = (\mathbb{C}\backslash D) \cup \{\infty\}$. Show that Ω is conformally equivalent to the upper half plane by explicitly finding a conformal map (or composition thereof) from Ω to the upper half plane.

4. Suppose $f(z)$ is analytic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $f'(0) = 0$. Show that $f(z)$ is not one-to-one on $\mathbb{D}$.

5. Find all functions $f(z)$ which are analytic and bounded on $\mathrm{Im}\, z > 0$, continuous on $\mathrm{Im}\, z \ge 0$, and real on $\mathrm{Im}\, z = 0$.

6. Find an entire function $f(z)$ such that $\displaystyle\oint_{|z|=2} \frac{f(z)}{nz-1}dz = \frac{\sqrt[n]{e}}{n}$ for all $n \in \mathbb{N}$. How many such functions are there?

7. Suppose $f(z)$ is an analytic function on $\mathbb{C}\backslash\{0\}$ and $|f(z)| \le \dfrac{1}{\sqrt{|z|}}$ for $z \in \mathbb{C}\backslash\{0\}$. Show that $f(z)$ is identically zero.

8. Derive the inequality $|\sin(\pi z)| \ge 2|z|$ for $|z| \le \dfrac{1}{2}$.

   Hint: you may use the infinite product formula $\sin(\pi z) = \pi z \displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$.