COMPLEX ANALYSIS QUALIFYING EXAM
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Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) Let $H = \{z = x + iy \ : \ y > 0, x \in \mathbb{R}\}$ denote the upper half–plane. Determine the image of H under the map

$$z \to \frac{1}{z + 1 + i}$$

and sketch the image.

2) Let $a_n, b_n \in \mathbb{C}$ denote two convergent sequences,

$$a_n \to a, \quad b_n \to b \ ,$$

and set

$$c_n = \frac{1}{n+1}\Big(a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0\Big) \ .$$

Prove that $c_n \to ab$.

Hint: First assume that $a_n = 1$ for all n and $b_n \to 0$.

3) Let $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$ denote the open unit disk. For $a \in \mathbb{C}$ with $|a| < 1$ define

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z} \quad \text{for} \quad z \in D \ .$$

Prove that the mapping $\phi_a : D \to D$ is one–to–one and onto.

Hint: Show that $\phi_a \circ \phi_{-a}$ is the identity on D.

4) Let Γ denote the positively oriented unit circle. Evaluate

$$\int_\Gamma \frac{dz}{z^5 + 3z^2 + 5} \ .$$

5) Determine the radius of convergence of the power series $\displaystyle\sum_{j=0}^{\infty} a_j z^j$ where

$$a_j = j^2 \Big(\frac{2 - 7j^4}{5j^4 - 2j + 1}\Big)^{3-j} \ .$$

6) Suppose $f(z)$ is an entire function and satisfies $\left|f\left(\frac{1}{\log(n+2)}\right)\right| < \frac{1}{n}$ for all integers $n \geq 1$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

7) Find all entire functions $f(z)$ such that $|f(z)| \geq \dfrac{1}{1 + |z|^{2015}}$ for all $z \in \mathbb{C}$.

8) Show that there exists an entire function $f(z)$ such that $f(n) = n^2$ at every positive integer n and $f(n) = -n^2$ at every negative integer n. How many such entire functions are there?