COMPLEX ANALYSIS QUALIFYING EXAM
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Department of Mathematics and Statistics
University of New Mexico

Instructions: Complete all problems to get full credit. Start each problem on a new page, number the pages, and put only your banner identification number on each page. Use only one side of each sheet.

Clear and concise answers with good justification will improve your score.

1) Decide whether the following statements are true or false. Provide a brief justification of your answer (e.g., quote a theorem or give a counterexample).
   a) $\lim_{z\to\infty} e^{-z} = 0$.
   b) $\cos(z) = \mathrm{Re}(e^{-iz})$ for all $z \in \mathbb{C}$.
   c) $|1 - z + z^2|$ is an entire function.
   d) $|z|^2 - 2z\mathrm{Re}(z)$ is an entire function.
   e) Every entire function with finitely many zeros is a polynomial.

2) a) Prove that the series

$$f(z) = \sum_{n=1}^{\infty} e^{-n} \sin(nz)$$

defines a holomorphic function in the strip

$$-1 < \mathrm{Im}(z) < 1 \ .$$

b) Does $f(z)$ have an analytic continuation outside the strip $-1 < \mathrm{Im}(z) < 1$? In which maximal domain?

Hint: Evaluate the series.

3) a) Let

$$f(z) = \frac{e^z}{(z-1)^4}, \quad z \in \mathbb{C} \setminus \{1\} \ .$$

Determine the Laurent expansion of f centered at $z = 1$.

b) Let Γ denote the positively oriented circle of radius 2 centered at $z = 0$. Determine

$$\int_\Gamma f(z)\, dz \ .$$

4) Find all entire functions $f(z)$ that satisfy $|zf(z) - 3 + e^{2z}| \le 4 + |z|$ for all $z \in \mathbb{C}$.

5) Find how many solutions of $z^3 - 3z^2 + 3 = 0$ are in the disk of radius 1, centered at 1.

6) Find a Möbius transformation that simultaneously maps the unit circle and the line $y = x$ onto the coordinate axes $x = 0$ and $y = 0$.

7) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of entire functions which converge uniformly on the unit circle $|z| = 1$. Show that there is a function f analytic on the unit disk $|z| < 1$ and such that $\lim_{n\to\infty} f_n(z) = f(z)$ for all $|z| < 1$.

8) Show that there is a meromorphic function f such that f has a simple zero at the origin with $f'(0) = 1$, a simple pole at $z = n$ with the residue equal n for each positive integer n, and no other poles.