Complex Analysis Qualifying Examination

August 2002 SS # : _____

Directions:

Do the following 8 problems. You may choose to answer the problems in any order. Please start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your SS #* in each page. Please show all your work and explain all steps in a proof or derivation.

Questions:

1. a) Three points, z_1, z_2, and z_3 satisfy the conditions

$$z_1 + z_2 + z_3 = 0, \text{ and } |z_i| = 1, \quad 1 \leq i \leq 3.$$

Show that these points lie at the vertices of an equilateral triangle inscribed in the unit circle.

b) Give a generalization (without proof) of the result in part (a) for the case of n points.

2. a) Give a precise statement of the Cauchy Integral Formula.
b) Give a proof of this formula.

3. a) State Rouche's theorem.
b) Find the number of zeros of the function

$$f(z) = 2z^5 + 7z^3 + z^2 - 3$$

in the annulus $1 < |z| < 2$

4. Let f(z) be a complex valued continuous function on a simple contour γ and define a function g(z) by

$$g(z) = \int_\gamma \frac{f(\xi)\cdot d\xi}{\xi - z}$$

a) Show that g(z) is analytic in any domain containing no points of γ.
b) Find an expression for g'(z).

5. a) Find a bound for the modulus of the integral shown below:

$$\int_\gamma \sin^2(z)\, dz$$

where γ is the simple contour $\gamma(t) = (1-t) + i\pi t$ and $0 \leq t \leq 1$.

b) Evaluate exactly the modulus of the integral in (a).

6. Classify the singularities and find residues for each of the following functions. Include points at infinity.

a) $z^2 + 2z^5$

b) $\dfrac{1}{\sin(z) - \cos(z)}$

c) $\cot(z)$

d) $\dfrac{1}{z\{\exp(z) - 1\}}$

7. Evaluate the integral

$$\int_0^{\pi} \frac{d\theta}{a + \sin^2(\theta)} \qquad a > 0$$

8. Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)\, dx}{1 + x^4}$$

Include justifications for all steps in your calculation.