Complex Variables Master's Qualifying Examination
January 1996

1) (a) Determine the region of absolute convergence of the products:

(i) $\prod_{n=1}^{\infty}(1-z^n)$

(ii) $\prod_{n=1}^{\infty}(1+z^{2n})$

(iii) $\prod_{n=0}^{\infty}(1+c_n z)$ where $\sum_{n=0}^{\infty}|c_n| < \infty$

(iv) $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^z}\right)$

(b) For the following functions in the <u>extended</u> complex plane give the branch points and the isolated singular points. Say whether the isolated singular points are removable, essential or poles (and, if poles, give the order)

(a) $\sqrt{z^2-1}$

(b) $\exp\left(\frac{1}{\sin z}\right)$

(c) $\ln(\sqrt{z^2+1})$

(d) $\tan z$

2) Let $D = \{z \in \mathbf{C} \mid |z| \leq 1 \text{ or } |z-2| \leq 1\}$. Let $\bar{D} = (\mathbf{C}\backslash D) \cup \{\infty\}$. Show $\bar{D}$ is conformally equivalent to the open upper half plane.

3) (a) Let f and g be entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| \geq 100$. Assume g is not identically zero. Show f/g is rational.

(b) Let u be harmonic in $\mathbf{C}$ and $u(x,y) \geq -2$ for all $x+iy \in \mathbf{C}$. Show u is constant in $\mathbf{C}$.

4) Let $f(z) = z^4 - 5z + 1$

(a) How many zeros does $f(z)$ have in the disc $\{z \in \mathbf{C} \mid |z| < 1\}$.
(b) How many zeros does $f(z)$ have in the annulus $1 < |z| < 2$.

5) Compute $F(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \dfrac{e^{ikx}}{x^2+1}\, dx$ where $k \in \mathbf{R}$

6) Expand (if possible) in Laurent series in the indicated region

(a) $e^{1/(z-1)}$ $|z| > 1$

(b) $\dfrac{1}{(z-a)(z-b)}$ (i) $0 < |a| < |z| < |b|$

                               (ii) $|a| < |b| < |z|$

(c) $\log\left(\dfrac{1}{1-z}\right)$ $|z| > 1$

7) Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\, dx$$

8) Choose a branch of $\sqrt{z^2 - 1}$ that is analytic on $\mathbb{C}\backslash\{x + 0i \mid -1 \le x \le 1\}$ and has the value $\sqrt{3}$ at $z = 2$. Evaluate $\int_\gamma \sqrt{z^2 - 1}\, dz$ where γ is the circle of radius 2, centered at 0 and oriented counterclockwise.