

MCTP - Summer Program 2010 (sponsored by NSF)

Minicourse: FOURIER ANALYSIS AND WAVELETS

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Project #4: **The Haar system in \mathbb{R}**

Alfred Haar constructed the Haar system on $[0, 1]$ in his PhD dissertation in 1909 (under Hilbert). One of the goals was to find an orthonormal system that will give uniform convergence of the partial sums for continuous functions, unlike the partial Fourier sums which could even diverge at some points, let alone converge uniformly. In this project we study the Haar basis in $L^2(\mathbb{R})$, and the multiresolution structure it encodes. First some definitions.

Let I be an interval, I_l and I_r the left and right halves of I (I 's children). The Haar function associated to the interval I is the following step function,

$$h_I(x) := \frac{1}{\sqrt{|I|}} (\chi_{I_r}(x) - \chi_{I_l}(x)) = \begin{cases} \frac{1}{\sqrt{|I|}} & \text{if } x \in I_r \\ \frac{-1}{\sqrt{|I|}} & \text{if } x \in I_l \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathcal{D} denote the dyadic intervals, more precisely, $I \in \mathcal{D}$ if and only if there exists integers j, k such that $I = [2^{-j}k, 2^{-j}(k+1))$. We denote by \mathcal{D}_j those dyadic intervals of length 2^{-j} .

The Haar system is the collection of Haar functions associated to the dyadic intervals. It is an *orthonormal basis* or a *complete orthonormal system*.

In this project we will first verify that the Haar system is orthonormal.

What is a little more work is to verify that the system is complete, that is that any square integrable function f can be written as a superposition of the Haar functions,

$$f(x) = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle h_I(x), \quad (1)$$

where the equal sign here means equality in the L^2 -norm. An equivalent formulation of the completeness of the Haar system is that there is NO square integrable function other than the ZERO function that is orthogonal to ALL Haar functions.

We will not check here the completeness of the Haar system, however you can find a complete argument in the excerpts from Chapter 9 and 10 from my book with Lesley Ward that I am handing out to you.

The first part of the project is devoted to question the plausibility of the completeness of the Haar system.

The second part is devoted to discussing the multiresolution analysis induced by the Haar system.

The Haar functions are the oldest example of a wavelet basis, although at the time, the terminology had not been dreamed yet, nor had the myriad of applications either.

A *wavelet basis* is an orthonormal basis of $L^2(\mathbb{R})$ generated by translations and dilations of one function, the *mother wavelet* ψ . More precisely the family of functions

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k), \quad \text{for } j, k \in \mathbb{Z},$$

is an orthonormal basis of $L^2(\mathbb{R})$.

Needless to say ψ must be a very special function to be a mother wavelet. To finish we will walk through the MATLAB Wavelet Toolbox Demo to illustrate *The world according to wavelets* (paraphrasing the title of Barbara Burke Hubbard's book for the broad public which is a good introduction to wavelets). Wavelets have been very successful for many applications, notably in image and signal processing. In your homework you are invited to construct a two dimensional multiresolution analysis (MRA) from a one dimensional MRA. This construction is what widely used to analyze images. You are also asked to extend to three dimensions, this type of MRA is used to transmit videos in internet for example: two spacial dimensions and the third is time.

You can continue your exploration of the Haar basis online. Google Haar basis and you get 8,770,000 hits. If you google wavelets you get 2,830,000. If you only do Haar you get over 50 million, but there might be many Haar's unrelated to Alfred Haar and to the Haar basis, furthermore Alfred Haar did a lot more in mathematics, for example he also introduced the Haar measure that might contribute a lot to the google count.

Handout

- Excerpts form Chapters 9-10 from the manuscript from the book *Harmonic Analysis: form Fourier to Haar*, coauthored with Lesley Ward.

WARMUP: THE HAAR SYSTEM IS AND ORTHONORMAL SYSTEM.

1. Check that given two dyadic intervals I and J , then either I is a proper subset of J , or J is a proper subset of I , or $I = J$, or they are disjoint.
2. Check that the Haar system is an orthonormal system, check also that each Haar function has integral zero.
3. If $I \in \mathcal{D}$ there is a unique pair of integers, j, k , such that $I = [2^{-j}k, 2^{-j}(k+1))$, we could use these integers to index the Haar functions,

$$h_{j,k}(x) := h_I(x).$$

Write a formula for $h_{j,k}$ involving ONLY the dilation parameter j , the translation parameter k , and the Haar function associated to the interval $[0, 1)$.

PART I: IS THE HAAR SYSTEM COMPLETE?

1. Let us now play the Devil's advocate.

- (a) First, consider the function $f(x) = 1$. Then $\langle f, h_I \rangle = \int_{\mathbb{R}} h_I(x) dx = 0$ for all $I \in \mathcal{D}$. We have found a function that is orthogonal to all the Haar functions, so how can the system be complete?
- (b) Second, how can it be true that functions that have zero integral (the Haar functions) can reconstruct functions that do not have zero integral?

If the Haar system is complete, then for any $f \in L^2(\mathbb{R})$, equation (1) holds. Integrating on both sides and interchanging the sum and the integral, we see that

$$\begin{aligned} \int_{\mathbb{R}} f(x) dx &= \int_{\mathbb{R}} \sum_{I \in \mathcal{D}} \langle f, h_I \rangle h_I(x) dx \\ &= \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \int_{\mathbb{R}} h_I(x) dx \\ &= 0, \end{aligned}$$

where the last equality holds because the Haar functions have integral zero.

- (c) We seem to be implying that all square integrable functions must themselves have integral zero. But we know this is not true, why?
- (d) What's wrong? Perhaps the Haar system is not complete after all. Or is there something wrong in the above calculation?

2. Let us check that the Haar expansion of the characteristic function of the unit interval, $\chi_{[0,1]}$ converges pointwise to itself.
 - (a) There should be many zero Haar coefficients, identify the dyadic intervals whose corresponding Haar functions yield non-zero coefficients.
 - (b) Write down now the Haar expansion for $f = \chi_{[0,1]}$ and explicitly evaluate it at all points.
3. Here we will explore whether the partial Haar sums converge uniformly to $\chi_{[0,1]}$ or not.

In the previous part you should have been able to tag all intervals associated to contributing Haar functions with a parameter, say $n \geq 1$, so that the intervals are nested, $I_1 \subset I_2 \subset I_3 \subset \dots$. Identify those contributing intervals.

Let

$$f_N(x) = \sum_{n=1}^N \langle \chi_{[0,1]}, h_{I_n} \rangle h_{I_n}.$$

- (a) Evaluate $f_N(x)$ for all $x \in \mathbb{R}$. What happens when you evaluate $\int_{\mathbb{R}} f_N(x) dx$?
- (b) Does $f_N \rightarrow \chi_{[0,1]}$ converges to $f = \chi_{[0,1]}$ uniformly on \mathbb{R} ?
- (c) Is it true that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} f_N(x) dx = \int_{\mathbb{R}} \lim_{N \rightarrow \infty} f_N(x) dx?$$

- (d) But wait, doesn't uniform convergence guarantee the interchange of the limit and the integral? What happened?

There is a very important result interchanging limits and integrals on \mathbb{R} , the Lebesgue Dominated Convergence Theorem. This theorem states that if a sequence of (measurable) functions f_n converges pointwise almost everywhere to a function f AND there is an integrable function g that dominates pointwise all the functions, namely, $|f_n(x)| \leq g(x)$, then the interchange of limit and integral is legal, i.e. $\lim_{n \rightarrow \infty} \int f_n = \int f$.

In our case, the functions f_N are measurable (whatever that means) and converge pointwise to $\chi_{[0,1]}$. We also know that the interchange does not hold, so it better be that there is NO dominating integrable function g . That is the content of the following exercise.

4. Verify that if a non-negative function g dominates all the functions f_N then it can not be integrable. Hint: try to find a function g_0 that "sits below" all dominating functions g , now show that g_0 is not integrable.

PART II: HAAR MULTIREOLUTION ANALYSIS

Here I want to show you that there is a lot of structure encoded in a Haar decomposition of a function. This structure is known as a MULTIREOLUTION ANALYSIS (MRA). Such structure is shared by a large collection of bases, the WAVELET BASES. The definition in all its glory, is hard to swallow...

An *orthogonal multiresolution analysis* is a collection of closed subspaces $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ such that

1. $\mathbf{V}_j \subset \mathbf{V}_{j+1}$ for all $j \in \mathbb{Z}$ (nested)
2. $\bigcap_{j \in \mathbb{Z}} \mathbf{V}_j = \{0\}$ (trivial intersection),
3. $\bigcup_{j \in \mathbb{Z}} \mathbf{V}_j$ is dense in $L^2(\mathbb{R})$ (density in $L^2(\mathbb{R})$),
4. $f(x) \in \mathbf{V}_j$ if and only if $f(2x) \in \mathbf{V}_{j+1}$ (scaling property),
5. $f(x) \in \mathbf{V}_0$ if and only if $f(x - k) \in \mathbf{V}_0$ for any $k \in \mathbb{Z}$ (shift invariance),
6. There exists a *scaling function* $\varphi \in \mathbf{V}_0$ such that its integer translates, $\{\varphi(x - k)\}_{k \in \mathbb{Z}}$, form an orthonormal basis for \mathbf{V}_0 .

So we have this doubly infinity nested sequence of subspaces

$$\{0\} \subset \cdots \subset \mathbf{V}_{-2} \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \mathbf{V}_2 \subset \cdots \rightarrow L^2(\mathbb{R})$$

Note that the scaling function φ completely determines the MRA subspaces, not all functions φ are scaling functions for an MRA, this are very special functions and identifying them is exactly the problem.

In this part of the project we will study the Haar MRA.

1. Let $\varphi(x) = \chi_{[0,1]}$, we will show that it is the scaling function of an MRA.
 - (a) Who is \mathbf{V}_0 ? Who are the subspaces \mathbf{V}_j ?
 - (b) Are the subspaces \mathbf{V}_j nested?
 - (c) Is the intersection of the subspaces \mathbf{V}_j trivial? Why?
 - (d) Is their union of the subspaces \mathbf{V}_j dense in $L^2(\mathbb{R})$? Why? in the next part we explore this issue further.

This is the Haar MRA, it is not yet clear why Haar, hopefully the last problem (item 3.) will clarify that.

2. Consider now the function $\varphi(x) = \sqrt{2}\chi_{[1/4,3/4]}(x)$. Can this function be the scaling function of an MRA? How about $\varphi(x) = \frac{2}{\sqrt{3}}\chi_{[0,4/3]}(x)$?

3. Here we explore the idea of resolution at many scales in the Haar MRA...

- (a) I give you a function $f \in L^2(\mathbb{R})$, and your job is to locate a function in \mathbf{V}_j which is closest in the L^2 -norm to f , who would you guess is the function? Calculate the orthogonal projection of f onto \mathbf{V}_j . It will be very useful to have an orthonormal basis for \mathbf{V}_j . Do we have it?
- (b) Denote by $P_j f$ the orthogonal projection to \mathbf{V}_j . Who is a better approximation of f , $P_j f$ or $P_{j+1} f$? We are incrementally improving the resolution of our function as we move up in the subspaces (this is how the internet works... you first get a blurred image, and it incrementally improves).
- (c) The better resolution function $P_{j+1} f$ has more information than the worst resolution function $P_j f$. Can you devise an easy way to go from $P_{j+1} f$ to $P_j f$? In the process we are losing information, the *details* necessary to move from $P_j f$ to $P_{j+1} f$, can you identify them? See items (d) and (e).
- (d) Let $Q_j f = P_{j+1} f - P_j f$. Show that $Q_j f$ is in \mathbf{V}_{j+1} and is orthogonal to \mathbf{V}_j . Denote by \mathbf{W}_j the orthogonal complement of \mathbf{V}_j in \mathbf{V}_{j+1} , so that any function in \mathbf{V}_{j+1} has a unique representation as a function on \mathbf{V}_j plus a function on \mathbf{W}_j , so that we can write

$$\mathbf{V}_{j+1} = \mathbf{V}_j \oplus \mathbf{W}_j$$

- (e) Show that

$$Q_j f(x) = \sum_{I \in \mathcal{D}_j} \langle f, h_I \rangle h_I(x).$$

Conclude that the subspace \mathbf{W}_j is the closure of the linear span of the Haar functions indexed on the dyadic intervals in

$$\mathcal{D}_j = \{I = [k2^{-j}, (k+1)2^{-j}] = \{I \in \mathcal{D} : |I| = 2^{-j}\}$$

You might be ready to revisit the issue of the density of the union of the subspaces \mathbf{V}_j , with the added information that the Haar system is an orthonormal basis of $L^2(\mathbb{R})$.

Mallat's Theorem says that given an orthogonal MRA with scaling function ϕ one can always find a wavelet ψ such that the subspaces \mathbf{W}_j are the closure of the linear span of the functions $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$, and the whole family, indexed over $j, k \in \mathbb{Z}$ forms an orthonormal basis in $L^2(\mathbb{R})$. Furthermore this is constructive and there is a FAST WAVELET TRANSFORM whenever the scaling function (and hence the wavelet) have compact support. Such compactly supported and smooth wavelets were discovered by Ingrid Daubechies now at Princeton University (she was the first female hired as a full Professor in the Math Department!).

Homework

Here we try to construct a two dimensional MRA from a given one.

There is a standard procedure to construct bases in 2-D space from given bases in 1-D, the *tensor product*. In particular, given a wavelet basis $\{\psi_{j,k}\}$ in $L^2(\mathbb{R})$, the family of tensor products

$$\psi_{j,k;i,n}(x,y) = \psi_{j,k}(x)\psi_{i,n}(y), \quad j,k,i,n \in \mathbb{Z},$$

is an orthonormal basis in $L^2(\mathbb{R}^2)$. Unfortunately we have lost the multiresolution structure. Notice that we are mixing up scales in the above process, that is the scaling parameters i, j can be anything.

1. Show that if $\{\psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis $L^2(\mathbb{R})$, then the family of functions defined on \mathbb{R}^2 by

$$\psi_{m,n}(x,y) = \psi_m(x)\psi_n(y),$$

is an orthonormal family for $L^2(\mathbb{R}^2)$. What requires some more work is to check that the family is a basis in $L^2(\mathbb{R}^2)$.

2. What would the trigonometric basis in $L^2([-\pi, \pi]^2)$ be? How about the finite-dimensional trigonometric basis, say in two dimensions?

We would like to use this idea but at the level of the approximation spaces \mathbf{V}_j in the MRA. For each scale j , the family $\{\varphi_{j,k}\}_k$ is an orthonormal basis of \mathbf{V}_j . Consider the tensor products

$$\varphi_{j,k;n}(x,y) = \varphi_{j,k}(x)\varphi_{j,n}(y)$$

of these functions. Let \mathcal{V}_j be the closure in $L^2(\mathbb{R}^2)$ of the linear span of those functions, in other words

$$\mathcal{V}_j = \mathbf{V}_j \otimes \mathbf{V}_j := \left\{ f(x,y) = \sum_{n,k \in \mathbb{Z}} a_{j,n,k} \varphi_{j,k;n}(x,y) : \sum_{n,k \in \mathbb{Z}} |a_{j,n,k}|^2 < \infty \right\}.$$

Notice that we are not mixing scales at the level of the MRA. It is not hard to see that the spaces \mathcal{V}_j form an MRA in $L^2(\mathbb{R}^2)$, with scaling function

$$\varphi(x,y) = \varphi(x)\varphi(y).$$

Therefore the integer shifts $\{\varphi(x-k, y-n) = \varphi_{0,k;n}\}_{k,n \in \mathbb{Z}}$ form an orthonormal basis of \mathcal{V}_0 , consecutive approximation spaces are connected via scaling by 2 on both variables, and the other conditions are clear.

The orthogonal complement of \mathcal{V}_j in \mathcal{V}_{j+1} is denoted by \mathcal{W}_j . The space \mathcal{W}_j can be viewed as the direct sum of three tensor products, namely

$$\mathcal{W}_j = (\mathbf{W}_j \otimes \mathbf{W}_j) \oplus (\mathbf{W}_j \otimes \mathbf{V}_j) \oplus (\mathbf{V}_j \otimes \mathbf{W}_j).$$

- (c) Can you give a heuristic argument why this should be true? assume that direct sum and tensor product of subspaces obey the usual rules of arithmetic.

Therefore *three wavelets* are necessary to span the detail spaces:

$$\psi^d(x, y) = \psi(x)\psi(y), \quad \psi^v(x, y) = \psi(x)\varphi(y), \quad \psi^h(x, y) = \varphi(x)\psi(y),$$

where d stands for diagonal, v for vertical, and h for horizontal. The reason for these names is that each of the subspaces somehow favors details in those directions.

- (d) Describe a three-dimensional tensor product MRA. (These are useful for video compression.) How many wavelets will be needed?
- (e) Describe the scaling and wavelets corresponding to the two-dimensional Haar Basis. Draw pictures of the two-dimensional Haar wavelets and scaling function.
- (e) Describe in words the two-dimensional discrete Haar MRA.

This construction has the advantage that the bases are separable, implementing the fast two dimensional Haar transform is not difficult. In fact it can be done by successively applying the one-dimensional FHT. The disadvantage is that the analysis is very axis-dependent, which might not be desirable for certain applications.