MCTP - Summer Program 2010 (sponsored by NSF)
Minicourse: Fourier Analysis and Wavelets
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Project #1: **The Gibbs Phenomenon**

We have mentioned that if a function f is just a little better than continuous then the Fourier series of f converges uniformly on $[-\pi, \pi]$. In this worksheet, first we will prove this statement if f has two continuous derivatives. Tomorrow I will try to convince you that the Fourier series in this case converges uniformly to f.
What happens if f is not continuous but instead has a jump discontinuity at some point x_0? Examples show that for "reasonable functions" the Fourier series converges to the midpoint of the jump, one can make this much more precise, see for example Körner's Chapter 16 p.59-61.
We will see that for the *square wave* function, that a 'blip' or overshoot is visible in the graph of the partial sum $S_N(f)$ on either side of the discontinuity at zero, and that while the width of these blips decreases as $n \to \infty$, the height does not. The existence of these blips is known as the *Gibbs phenomenon.* In the remainder of this project we explore the Gibbs phenomenon for Fourier series.

1. Show that if f is periodic and has two continuous derivatives then the Fourier series of f converges *uniformly* on $[-\pi, \pi)$, in tomorrow's project I'll try to convince you that the Fourier series converges to f.

   (a) Use integration by parts to verify the following very useful formula:

   $$\widehat{f'}(n) = in\widehat{f}(n),$$

   knowing that f is 2π-periodic, and continuously differentiable, that is $f \in C^1(\mathbb{T})$. Use induction now to check that if $f \in C^k(\mathbb{T})$ (2π-periodic and k-continuous derivatives) then

   $$\widehat{f^{(k)}}(n) = (in)^k \widehat{f}(n).$$

   (b) What does the previous item tell you in terms of "smoothness" of f versus "decay" of the Fourier coefficients?

   (c) Invoke the M-Weierstrass test to show now that if $f \in C^2(\mathbb{T})$ then the partial Fourier sums of f converge "uniformly". Why can't you use this argument if all you know is $f \in C^1(\mathbb{T})$?

   The Fourier series converges uniformly also when $f \in C^1(\mathbb{T})$, and one can further relax this assumption. But it is a false statement for continuous functions, there is a periodic continuous function whose partial Fourier sums diverge at $x = 0$. See Chapter 18 from Körner's book.

2. If you know how to, use MATLAB to calculate the values of, and to plot, several partial Fourier sums $S_N g$ for the *square wave* function

$$g(x) := \begin{cases} 1 & \text{if } 0 \le x < \pi \\ -1 & \text{if } -\pi \le x < 0. \end{cases}$$

(Extend this function periodically and the name will explain itself.)

Concentrate on the blips near zero (you should see another blip near $\pm\pi$ as well). If you know how to do it, animate your plots to create a movie showing what happens as N increases. From your results, estimate the height of the blip. Is this height related to the size or location of the jump? If so, how? What happens near $x = 0$ to the x-location and the width of the blips as N increases?

3. Prove analytically that the height of the blip near $x = 0$ for the square wave function g does not tend to zero as $N \to \infty$. One approach is as follows.

   (a) Before you do anything, does the function have some symmetry? What does that tell you about the the zero-th Fourier coefficient and the Fourier series?

   (b) Calculate the Fourier coefficients $\widehat{g}(n)$ and the partial Fourier sums of g. Evaluate the Fourier series at $x = 0, \pm\pi$. Evaluate at $x = \pi/2$ and pretend you know that the series converges to $1 = g(\pi/2)$, what do you learn?

   ANS: $\widehat{g}(n) = 0$ if n is even, $\widehat{g}(n) = 2/\pi i n$ if n is odd. Write as a series in sines.

   (c) Get a closed formula (no sums involved) for the partial Fourier sums $S_{2N-1}(g)$, or for its derivative $[S_{2N-1}(g)]'$. Hint: In either case it might be useful to verify that

   $$\sum_{n=1}^{N} \cos\big((2n-1)x\big) = \frac{\sin(2Nx)}{2\sin x}.$$

   (d) Use calculus to find the critical points of $S_{2N-1}(g)$ (points where the derivative is equal to zero), and check that the point $x_N = \frac{\pi}{2N}$ is actually a local maximum. Can you identify a local minimum?

   (e) Verify that

   $$S_{2N-1}(g)(x_N) \sim \frac{2}{\pi}\int_0^{\pi} \frac{\sin u}{u}\, du.$$

   (f) Consider the 2π-periodic *sawtooth* or *ramp* function defined to be $f(x) = x$ for $-\pi \le x < \pi$. The discontinuity occurs now at odd multiples of π, some numerical exploration will reveal that the Gibbs phenomenon occurs, can you analitically prove it? See hand-outs.

(g) The two examples analyzed so far involve at the end the value

$$\mathrm{Si}(\pi) := \int_0^\pi \frac{\sin u}{u}\, du$$

Look up in a table, use Taylor series, or use MATLAB to calculate, the numerical value of the integral $\mathrm{Si}(\pi)$. ANS: $\mathrm{Si}(\pi)$ is approximately $1.17898(\pi/2)$. With this numerical information at hand, interpret your results to explain the blips in the previous examples.

(h) Can the graphs of the partial Fourier sums of g "converge" in any reasonable sense to the graph of g? Can we say something about the arclengths of the graphs Γ_N of $S_N g$? See hand-out by Strichartz. Here,

$$\Gamma_N = \{(x, S_N f(x)) : x \in [-\pi, \pi)\}.$$

ANS: NO, $\text{length}(\Gamma_N) \geq \log N$.

(i) Is there anything we can do to "cure" the Gibbs Phenomenon just discovered for g? Can we "tame down" the harmonic series that is lurking in the background by some sort of averaging?

Homework:

- We have worked out one example (square box function), and have literature on another (sawtooth function) and seen that they exhibit the Gibbs Phenomenon near $x = 0$ or $\pm\pi$. Numerical experiments will show that the phenomenon is present in many other functions that have jump discontinuities. To analytically prove the existence of the blips for each one of these examples, do we have to start all over? or, can we get mileage out of the work already done?

  1. Consider first step or linear periodic functions with a jump at some $x_0 \neq 0$.
  2. Consider next step or linear functions with discontinuities at several points.
  3. Can we now extend the results to the case of a "general" function with a jump discontinuity? Part of the exercise is to decide how "general" the functions can be (we will content ourselves with some degree of generality, not the most general class :-).

- Explore the literature related to the Gibbs phenomenon. Here are some references to get you started: Körner's *Fourier Analysis* Chapters 15-17, Cambridge University Press, 1988; Elena Prestini's *The evolution of harmonic analysis of the real world*, Birkhäuser, Boston, MA, 2004; and Gottlieb's and Shu's article *On the Gibbs phenomenon and its resolution*, SIAM Rev. 39, 644-668 (2005) give overviews of the history, while more detail is in Gibbs' two letters to *Nature* and Michelson's letter to the same journal, and in the papers by Wilbraham, Bôcher, and Michelson and Stratton.

There is a book completely dedicated to the subject *The Gibbs Phenomenon in Fourier Analysis, Splines and Wavelet*, By Abdul J. Jerri, Kluwer Academic Publishers (Springer), 1998. As well as a compendium of papers with the latest results *Advances in the Gibbs Phenomenon with detailed introduction*, edited by Abdul J. Jerri, Σ Sampling Publishing, 2007.

If you google Gibbs Phenomenon you get 193,000 results starting with Wikipedia.

Here is a pretty cool Applet showing Gibbs Phenomenon for the ramp function and for another discontinuous function,

http://ocw.mit.edu/ans7870/18/18.06/javademo/Gibbs/

I am giving you some handouts, in chronological order (thanks to Kourosh Raeen for saving me a trip to the library, his MS thesis was the inspiration for this project):

- Wilgraham's article *On a certain periodic function*, Cambridge and Dublin Math. J. 3 (1848), 198-201.
- Michelson and Stratton's article *A new harmonic analyser*, Philosophical Magazine, (1898) p. 85-91.
- Michelson's October 6, 1898 Letter to Nature, and Gibbs's December 29, 1898 and April 27, 1899 Letters to Nature.
- Carslaw's note on *A historical note on Gibbs' phenomenon in Fourier series*, and Moore's *Note on Gibbs' Pehnomenon*, Bull. Amer. Math. Soc. Volume 31, Number 8 (1925), 417-424.
- Chapters 16-19 from Körner's book, p. 59-75.
- R. Strichartz's article *Gibbs' Phenomenon and Arclength*, J. Fourier Anal. and Appl., Volume 6, Issue 5 (2000) 533-536.
- Section 2.5 from Prestini's book, *Lord Kelvin, Michelson, and Gibbs phenomenon*, p. 49-52.

Hopefully after reading about the history of the Gibbs phenomenon the following comment borrowed from the article *The Gibbs-Wilbraham Phenomenon: an Episode in Fourier analysis* by E. Hewitt and R. Hewitt will make sense:

> "GIBBS's phenomenon, while not a fundamental part of mathematics, displays *in parvo* a number of central features of the development of mathematics. We find forgotten pioneers. We encounter shocking dispute over priority. We study brilliant achievements, some (like WEYL and GRONWALL) never properly appreciated. We discover a remarkable succession of blunders, which could hardly have arisen save through copying from predecessors without checking.
>
> In short, GIBBS's phenomenon and its history offer ample evidence that mathematics, for all its majesty and austere exactitude, is carried on by humans."

In the Archive for History of Exact Sciences 21 (1979) no.2, pp. 129-160.