MCTP - Summer Program 2010 (sponsored by NSF)
Minicourse: Fourier Analysis and Wavelets
Instructor: Cristina Pereyra
University of New Mexico
Project #3: **The Fast Fourier Transform and the Fast Haar Transform**

In this project we capitalize on the knowledge acquired while studying Fourier series to develop the simpler Fourier theory in a finite-dimensional setting.
In this context, the *discrete trigonometric basis for* $\mathbb{C}^N$ is a collection of N orthonormal vectors, hence an orthonormal basis. The *discrete Fourier transform* is the linear transformation that gives us the coefficients in the trigonometric basis. Many of the features of the Fourier series theory are still present, but without the nuisances and challenges of being in an infinite-dimensional setting ($L^2(\mathbb{T})$), where one has to deal with integration and infinite sums. The tools required in the finite-dimensional setting are the tools of LINEAR ALGEBRA.
For many practical purposes, this finite theory is what is needed. Computers deal with finite vectors. The Discrete Fourier Transform (DFT) is calculated by applying an invertible $N \times N$ matrix to a given vector in $\mathbb{C}^N$, and the Discrete Inverse Fourier Transform by applying the inverse matrix to the transformed vector. The surprising fact is that one can perform these matrix multiplications in order $N \log_2 N$ operations, as opposed to the expected N^2 operations, using the celebrated *Fast Fourier Transform* (FFT). This gain in the number of operations is of invaluable importance for numerical applications, especially when dealing with large problems.
We introduce another orthonormal basis, the *discrete Haar basis*. We highlight the similarities and the differences with the discrete trigonometric basis. There is a *Fast Haar Transform* (FHT), of order N operations. The Fast Haar Transform is a precursor of the *Fast Wavelet Transform* (FWT).
The goals of this project is to understand the FFT and the FHT algorithms. These are two examples of a very useful strategy: *divide and conquer*.
To continue your exploration of the FFT, you can of course make an online search. A Google search for FFT gives 7,400,000 quotes!!!!
The book by Michael Frazier, *An Introduction to Wavelets through Linear Algebra*, Springer-Verlag (1999), contains in Chapters 2 and 3, a very careful description of the discrete Fourier transform, and the discrete Wavelet transform (in particular the discrete Haar transform), using just the tools of linear algebra.

Handouts

- The 1965 Cooley and Tukey's paper, *An Algorithm for the Machine Calculation of Complex Fourier Series.* Mathematics of computation, Vol. 19, No. 90, (Apr., 1965), pp. 297-301.
- David Rockmore's article *The FFT - an algorithm the whole family can use.* http://www.cs.dartmouth.edu/~rockmore/fft.html
- Chapter 6 from my book with Lesley Ward.
- From Prestini's book, Section 3.10 *Gauss and the asteroids: history of the FFT.*

1. **Dual Bases:** This is a parenthesis on bases and dual bases in $\mathbb{C}^N$, a review of linear algebra.

   (a) (warmup) Consider the vectors $\vec{v_1} = (1,0)$ and $\vec{v_2} = (1,1)$ in $\mathbb{R}^2$. Are they linearly independent? Are they orthogonal?

   Write the vector $\vec{v} = (2,3)$ as a linear combination of $\vec{v_1}$, $\vec{v_2}$

   Show that any other vector $\vec{v} = (x_1, x_2)$ in $\mathbb{R}^2$ can be writen as a linear combination of $\vec{v_1}$ and $\vec{v_2}$, that is, there exist unique real numbers a_1 and a_2 such that

   $$\vec{v} = a_1\vec{v_1} + a_2\vec{v_2}.$$

   Can you find *dual vectors* $\vec{w_1}$ and $\vec{w_2}$ so that the coefficients are found by taking the dot product of the given vector and the dual vectors, that is $a_i = \vec{v} \cdot \vec{w_i}$ for $i = 1,2$? Do these vectors have any special geometric property vis a vis the given vectors $\vec{v_1}$ and $\vec{v_2}$? Verify that the roles of $\{\vec{w_1}, \vec{w_2}\}$ can be interchanged, more precisely check that:

   $$\vec{v} = \langle \vec{v}, \vec{w_1} \rangle \vec{v_1} + \langle \vec{v}, \vec{w_2} \rangle \vec{v_2} = \langle \vec{v}, \vec{v_1} \rangle \vec{w_1} + \langle \vec{v}, \vec{v_2} \rangle \vec{w_2}.$$

   (b) Given N linearly independent vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_N}\}$ in $\mathbb{C}^N$, in other words a basis for $\mathbb{C}^N$, let B be the invertible matrix with columns $\vec{v_j}$, and denote by $\vec{w_j}$ the complex conjugate of the jth row vector of its inverse B^{-1}.

      i. Verify that the jth entry of the vector $B^{-1}\vec{v}$ is the (complex!) inner product between the vector $\vec{v}$ and the vector $\vec{w_j}$, and it is also the jth coefficient a_j in the expansion of the vector $\vec{v}$ in the given basis, that is

      $$\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_N\vec{v_N} = \sum_{j=1}^{N} a_j\vec{v_j}.$$

      So we got a formula for the coefficients!!! Verify that the roles of $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_N}\}$ and $\{\vec{w_1}, \vec{w_2}, \ldots, \vec{w_N}\}$ can be interchanged, justifying the name *dual bases*.

      ii. Is there any interesting relation between the vectors $\{\vec{v_j}\}$ and the *dual vectors* $\{\vec{w_j}\}$? (Hint: remember that B and B^{-1} are encoded in those vectors.) The fact that the roles of $\{\vec{v_j}\}$ and $\{\vec{w_j}\}$ can be interchanged can be restated in the language of matrix theory: existence of a right (or left) inverse guarantees existence of an inverse for $N \times N$ matrices.

      iii. Can the vectors $\vec{w_j} = \vec{v_j}$ for all $j = 1, \ldots, N$? If yes, what special property should the vectors satisfy? What special shape will the matrix B^{-1} have in that case? Do these matrices have a special name?

2. **Fast Fourier Transform:** We now explore the discrete Fourier basis in $\mathbb{C}^N$.

   (a) (warmup) Consider the vectors f_0, f_1, f_2, f_3, in $\mathbb{C}^4$. Denote $f_n(k)$ the kth entry of the nth vector, where

   $$f_n(k) = e^{2\pi ikn/4}, \quad \text{where } k, n = 0, 1, 2, 3.$$

   Let F_4 be the 4×4 matrix whose columns are the vectors $\{f_n\}$ for $n = 0, 1, 2, 3$. (Notice that the dimension $N = 4$ shows in the denominator of the exponent, we should tag the vectors with N indicating in what dimension we are, for example f_n^N, in this case $f_n^N(k) = e^{2\pi ikn/N}$. I have not done such labeling, however if you feel more confortable writing the superscript N, please by all means do it.)

      i. Verify that the vectors $\{f_n\}$ for $n = 0, 1, 2, 3$ are orthogonal vectors in $\mathbb{C}^4$. Compute their lengths. Are they normalized? If not, is it hard to normalize them? Denote by $\{e_n\}$ the normalized vectors, this is the *discrete Fourier basis* in $\mathbb{C}^4$. Write out explicitely the vectors $\{e_0, e_1, e_2, e_3\}$.

      ii. What does the previous item tell us about the matrix F_4? Find the inverse of F_4. How do we find the coefficients of a vector in $\mathbb{C}^4$ with respect to the discrete Fourier basis in $\mathbb{C}^4$ in terms of the matrix F_4?

      iii. Do you see a way of factorizing the matrix F_4 as a product of two 2×2 block matrices with blocks of dimensions 2×2? (Hint: you would very much like to see F_2 among the blocks.)

   (b) Can you now repeat the above steps for any dimension N? This time assume $N = 2M$, you are seeking to decompose F_{2M} as a product of 2×2 block matrices with blocks of dimensions $M \times M$? (Hint: you would very much like to see F_M among the blocks.)

   (c) The problem of computing the discrete Fourier coefficients of a vector in $\mathbb{C}^N$ reduces to applying a multiple of the complex conjugate of the matrix F_N.

      i. The matrix F_N is a full $N \times N$ matrix. Applying the matrix $\overline{F_N}$ as it is to a vector in $\mathbb{C}^N$ involves how many complex multiplications?

      ii. If $N = 2M$ and you take advantage of the discovered block structure how many multiplications will be needed. If $N = 2^J$, you can now iterate on the smaller dimensional Fourier matrices that appeared in the block decomposition. How much can you iterate, and if you iterate as much as you can, how many multiplications will be involved in the calculation?

3. **Fast Haar Transform:** We now explore the discrete Haar basis in $\mathbb{R}^N$.

(a) (warmup) Let $N = 2^3 = 8$. Consider the vectors $\tilde{h}_n$ in $\mathbb{R}^8$ for $n = 0, 1, \ldots, 7$, defined by:

$$\begin{aligned}
\tilde{h}_0 &:= [1, 1, 1, 1, 1, 1, 1, 1], \\
\tilde{h}_1 &:= [-1, -1, -1, -1, 1, 1, 1, 1], \\
\tilde{h}_2 &:= [-1, -1, 1, 1, 0, 0, 0, 0], \\
\tilde{h}_3 &:= [0, 0, 0, 0, -1, -1, 1, 1], \\
\tilde{h}_4 &:= [-1, 1, 0, 0, 0, 0, 0, 0], \\
\tilde{h}_5 &:= [0, 0, -1, 1, 0, 0, 0, 0], \\
\tilde{h}_6 &:= [0, 0, 0, 0, -1, 1, 0, 0], \\
\tilde{h}_7 &:= [0, 0, 0, 0, 0, 0, -1, 1].
\end{aligned}$$

i. Verify that the vectors $\{\tilde{h}_n\}$ for $n = 0, 1, \ldots, 7$ are orthogonal vectors in $\mathbb{R}^8$. Compute their lengths. Are they normalized? If not, is it hard to normalize them? Denote by $\{h_n\}$ (h for Haar) the normalized vectors, they form the discrete Haar basis for $\mathbb{R}^8$. Write out explicitely the normalized vectors $\{h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$.

Let H_8 be the matrix whose columns are the vectors $\{h_n\}$ for $n = 0, 1, \ldots, 7$.

ii. What does the previous item tell us about the matrix H_8? Find the inverse of H_8. How do we find the coefficients of a vector in $\mathbb{R}^8$ with respect to the discrete Haar basis in $\mathbb{R}^8$ in terms of the matrix H_8?

iii. Do you see a way of factorizing the transpose matrix H_8^t as a product of two sparser block matrices? (Hint: you would very much like to see H_4^t and H_2^t among the blocks. Consider first a decomposition of H_4^t)

(b) Can you now repeat the above steps for any dimension N? This time assume $N = 2^J$, you are seeking to decompose H_N as a product of two sparser block matrices. (Hint: you would very much like to see $H_{N/2}$ among the blocks.)

(c) The problem of computing the discrete Haar coefficients of a vector in $\mathbb{C}^N$ reduces to applying the transpose H_N^t.

i. The matrix H_N is not a full $N \times N$ matrix. How many entries are non-zero? Applying the matrix H_N^t as it is to a vector in $\mathbb{C}^N$ involves how many complex multiplications?

ii. Since $N = 2^J$ and you take advantage of the discovered block structure how many multiplications will be needed. You can now iterate on the smaller dimensional Haar matrices that appeared in the block decomposition. How much can you iterate, and if you iterate as much as you can, how many multiplications will be involved in the calculation?

Homework: Bit Reversal

In the factorization leading to the FFT, after iterating, a product of permutation matrices appeared. Here we explore a litle numerical miracle.
First a bit of notation.
Let the $M \times 2M$ matrices Even_M and Odd_M that grab in order the even entries and the odd entries of a given $2M$-vector respectively (with the convention that the vector is indexed starting at zero).

1. Who are the matrices Even_2, Odd_2, Even_4 and Odd_4.

2. Define the 8×8 scrambling matrices by

$$S_1^8 := \begin{bmatrix} \text{Even}_4 \\ \text{Odd}_4 \end{bmatrix}, \quad S_2^8 := \begin{bmatrix} \text{Even}_2 & \\ \text{Odd}_2 & \\ & \text{Even}_2 \\ & \text{Odd}_2 \end{bmatrix}.$$

   The empty spaces correspond to zero matrices of the appropriate dimension (in this case 4×4).

   Describe in words what is the action of matrix S_2^8 on a vector $v \in \mathbb{C}^8$.

3. Consider the vector $v = (0, 1, 2, 3, 4, 5, 6, 7)$, compute $S_2^8 S_1^8 v$.

4. Write both the entries of v and the entries of $S_2^8 S_1^8 v$ in binary notation. Do you observe something peculiar? Denote the “miracle matrix” in dimension $2^3 = 8$ by $M_3 = S_2^8 S_1^8$.

5. Do you think the same miracle will occur when $N = 2^j$? Can you justify it? Try an argument by induction, notice that the corresponding miracle matrix in dimension $N = 2^j$ is the product of $j - 1$ scrambling matrices, $M_j = S_{j-1}^N \dots S_2^N S_1^N$.

Homework: Circular Convolution vs Discrete Fourier Transform

Given two vectors $v, w \in \mathbb{C}^N$, we define their *circular convolution* $v * w \in \mathbb{C}^N$ by

$$v * w(n) = \sum_{k=0}^{N-1} v(k)w(n-k),$$

where the vector w has been extended periodically with period N, that is, $v(n) = v(n+N)$ for all $n \in \mathbb{Z}$. Let $\widehat{v} = \frac{1}{\sqrt{N}}\overline{F_N}v$ be the *discrete Fourier transform* of v, that is $\widehat{v} \in \mathbb{C}^N$, with entries

$$\widehat{v}(n) = \langle v, e_n \rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k) e^{2\pi i n k/N}$$

Show that

$$\widehat{v * w}(n) = \widehat{v}(n)\widehat{w}(n).$$

(We may need a constant in the formula for the convolution to get this last one without a constant.) Compare to Project #2 Homework: *Convolution is a smoothing operation*, Problem 2.

Show that this formula guarantees that we can perform circular convolutions in $\mathbb{C}^N$ in order $N \log_2 N$ operations as well (vs N^2).