

Groups, isometries, and special relativity

1. The collection of isometries of the Euclidean plane, which we will denote by G , is a group. This means that there is a notion of *composition* of isometries: that is if f, g are isometries of the Euclidean plane we can make sense of fg and gf (here, the natural interpretation of fg and gf is *composition of maps*). Moreover we have the following three properties of the set of isometries of the Euclidean plane:
 - i. Composition is associative, that is $(fg)h = f(gh)$ for any three isometries f, g, h .
 - ii. There is an identity isometry e , that is an isometry with the property that $fe = ef = f$ for any isometry f .
 - iii. Given an isometry f there is a unique isometry f^{-1} with the property that $ff^{-1} = f^{-1}f = e$.

Find the identity isometry e , the inverse f^{-1} for an arbitrary isometry f , and verify that **i**, **ii**, and **iii** are satisfied.

2. Each of the following is a collection of isometries of the euclidean plane. For each collection, determine whether or not the group properties of problem 1 above are satisfied. Explain.
- a. $SO_2(P) = \{\text{rotations about a point } P \in \mathbf{R}^2\}$,
 - b. $O_2(P) = \{\text{all isometries fixing a point } P \in \mathbf{R}^2\}$,
 - c. $R_2 = \{\text{all rotations}\}$,
 - d. $T_2 = \{\text{all translations}\}$,
 - e. $G_2 = \{\text{all glide reflections}\}$,
 - f. $S_2 = \{\text{all reflections}\}$.

3. A very different and very important notion of symmetry arises in relativistic mechanics and we will briefly discuss this here. The special theory of relativity indicates that the laws of physics governing our world do not change in a *constantly moving framework*; alternatively
- There is no natural absolute spatial coordinate system with respect to which we can determine motion.
 - No experiments can be devised whose results would establish such an absolute coordinate system.

One “corollary” of these principles is that the speed of light, denoted c , must be the same measured within any moving framework (otherwise, if this speed depended on the framework, it could be used to detect uniform motion). For the special theory of relativity, then, there are many possible spatio-temporal frameworks, each governed by the same laws of physics, and one should expect a simple means of passing from one framework to another. These are the so-called *Lorentz transformations* which we will try to derive here. For this problem we will work in one spatial dimension and also add a temporal dimension. For the first spatio-temporal coordinate system we take the x -axis with temporal coordinate t . For the second spatio-temporal coordinate system (x', t') we will assume that at $t = t' = 0$ the origin of the x and x' axes correspond to one another and we will assume that the (x', t') system is moving with constant velocity v relative to the (x, t) system.

- Find x' as a function of x , *assuming that $t = t'$* .
- Suppose that a beam of light is emitted from $x = -1$, going in the direction of the positive x -axis, at $t = 0$. Suppose c is the speed of light relative to the fixed coordinate system (x, t) . What will the perceived speed of the light be by a person using coordinate system (x', t') , still assuming that $t = t'$?
- Suppose Alice, using coordinate system (x', t') , measures the velocity of the light beam and finds that it is c . What might she conclude?
- Can you think of any natural explanation why the speed of light might be the same regardless of the (constantly) moving framework in which it is measured?

4. In this problem we will study the relationship between the two coordinate systems (x, t) and (x', t') .
- In relating (x', t') and (x, t) there are two factors to consider. First at time t in the (x, t) system, the (x', t') system has travelled a certain distance relative to the (x, t) system and this displacement needs to be subtracted from x in order to obtain x' . Find this displacement (this is problem **3a**).
 - Secondly, and more subtly, there is a scaling factor in order to relate units of distance/time in the (x', t') system to the equivalent units in the (x, t) system. One way to see (and compute) the scaling factor is this: imagine a train moving with constant velocity v . Take for the coordinate system (x, t) the train tracks and for coordinates (x', t') the train. Imagine that a beam of light is emitted from the floor of the train and reflected back from the ceiling. In the (x', t') coordinate system the path of the light will be vertical, both on the way up and on the way down. In the (x, t) system it will not be. Calculate the time the light takes to traverse this path in the (x', t') coordinate system. Next calculate the time in the (x, t) system (of course these calculations will rest on the assumption that the speed of the light is c in both frameworks). Find the scaling factor which relates units of time in the two coordinate systems. Because c is a rate of change, the same scaling factor relates units of distance in the two frameworks. The final coordinate changes you should find is

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- For time, that is to relate t' to t , there is a scaling factor which you have already found in part **b**: why is the scaling factor the same for time and distance? There is, however, also a *spatial* component: why? The appropriate formula is

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

5. The transformations you found in Problem 4 which relate the spatial and temporal coordinates of constantly moving systems are known as the Lorentz transformations. They only make sense if $-c < v < c$ (unless you are on the Starship Enterprise!). We only dealt with a single spatial coordinate but of course similar transformations exist for four dimensional space–time with coordinates (x, y, z, t) . Endowed with the set of Lorentz transformations, this is called *Minkowski space–time*.
- a. Suppose you have three coordinate systems (we will return to $((x_1, t_1), (x_2, t_2), (x_3, t_3)$ for simplicity). Suppose the second has a velocity of v_1 relative to the first and the third has a velocity of v_2 relative to the second. What is the relationship between the third coordinate system and the first? (Be careful!!!).
 - b. Show that the Lorentz transformations form a group of symmetries of Minkowski space–time.