

Leonardo's Theorem

1. Suppose that l is a line in \mathbf{R}^2 and $v \in \mathbf{R}^2$ a non-zero vector so that the line containing v is parallel to l . The map $\gamma_{l,v} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ given by

$$\gamma_{l,v}(x) = \sigma_l(v + x)$$

is called a *glide reflection*: here σ_l denotes reflection about the line l .

- a. Suppose that l is the x -axis and $v = (5, 3)$. Describe the effect of the glide reflection $\gamma_{l,v}$ in this case.
- b. Can you find different vectors v' and/or different lines l' so that $\gamma_{l',v'} = \gamma_{l,v}$, where $\gamma_{l,v}$ is the symmetry of part **a**?
- c. Do glide reflections ever have fixed points?

- 2.** Suppose $k, l, m \subset \mathbf{R}^2$ are lines. Here we will study the composition of the three reflections $\sigma_k, \sigma_l, \sigma_m$ in some degenerate situations.
- a.** Suppose that k, l, m are parallel and not necessarily distinct. Identify the isometry $\sigma_k \circ \sigma_l \circ \sigma_m$.
- b.** Suppose that k, l, m all meet at a common P but share no other common point in common. What is $\sigma_k \circ \sigma_l \circ \sigma_m$ in this case?

- 3.** Now we will treat the case where $k, l, m \subset \mathbf{R}^2$ are three lines, no two of which are parallel.
- a.** Suppose that m is the x -axis, l is the y -axis, and k is the line $x + y = 1$. Find $\sigma_k \circ \sigma_l \circ \sigma_m$ in this case.
 - b.** What can you conclude about $\sigma_k \circ \sigma_l \circ \sigma_m$ in general?

4. Suppose G is a *finite* set of isometries of \mathbf{R}^2 which is *closed* under composition (that is, if $f_1, f_2 \in G$ then $f_1 \circ f_2$ is also in G). Show the following:
- a. There are no translations (other than the identity) in G .
 - b. There are no glide reflections in G (except those where the translation is by $\vec{0}$, that is unless the glide reflection is a reflection).
 - c. If G contains any rotations, then all of these rotations have the same center.

5. For this problem G is the same as in Problem 4.

- a. Using the classification of symmetries from the previous problem set, show that G can only contain reflections and rotations.
- b. Show that if G contains rotations then the full set of rotations in G has a common center P . Moreover, there is a positive integer n so that the degrees of the rotations in G , about P , are $\frac{2\pi i}{n}$ for $0 \leq i < n$.
- c. Show that if G contains non-trivial reflections, then these reflections are all of the form $\rho \circ r$ where ρ is a rotation (in G) about P and r is a reflection about a line L which contains P .
- d. Conclude that G is *either* a set of rotations as in **b** *or* G is of the form

$$\{\rho_0, \dots, \rho_{n-1}, r\rho_0, \dots, r\rho_{n-1}\}$$

where ρ_i denotes the rotation through $\frac{2\pi i}{n}$ degrees about P and r is the reflection about L , a line containing P .

- e. For each possible finite set G of symmetries as above, give a geometric figure for which G is the *full* set of symmetries.
- f. For some small simple finite sets of symmetries G considered in this problem set, can you find molecules for which G is the full set of symmetries?