# Galois theory and chemistry

The collection of roots of polynomials display many types of symmetry. The theory which studies these symmetries is called Galois theory and it is a very technical, difficult, but beautiful subject. To a given polynomial $f(X)$ of degree $d$ with integer (or rational) coefficients is associated a collection $G_f$ of symmetries of the roots called the Galois group of $f$. If $\{\alpha_1, \ldots, \alpha_d\}$ are the roots of $f$ and $\sigma \in G_f$ then $\sigma$ *permutes* the roots of $f$, that is

$$\{\sigma(\alpha_1), \ldots, \sigma(\alpha_d)\} = \{\alpha_1, \ldots, \alpha_d\}.$$

Not all permutations are allowed, however, only those which preserve the "algebraic structure" of the roots, meaning that if $P(\alpha_1, \ldots, \alpha_d)$ is some polynomial expression involving the roots $\alpha_1, \ldots, \alpha_d$, with integer coefficients, then

$$P(\alpha_1, \ldots, \alpha_d) = 0 \iff P(\sigma(\alpha_1), \ldots, \sigma(\alpha_d)) = 0.$$

**1.** To get a feel for which permutations are in the group $G_f$ associated to a polynomial $f$ consider the following simple cases.

**a.** Suppose $f(X) = X^2 - 2$. Show that the non–trivial permuation is in $G$, that is show that if $P(X, Y)$ is a polynomial in two variables with integer coefficients then

$$P\left(\sqrt{2}, -\sqrt{2}\right) = 0 \iff P\left(-\sqrt{2}, \sqrt{2}\right) = 0.$$

**b.** Suppose $g(X) = (X^2 - 2)(X^2 - 3)$. Show that it is *not* possible to interchange $\sqrt{2}$ and $\sqrt{3}$, that is there is no element $\sigma \in G_g$ with $\sigma(\sqrt{2}) = \sqrt{3}$ and $\sigma(\sqrt{3}) = \sqrt{2}$. What is $G_g$?

**2.** Consider $h(X) = X^3 - 2$.

**a.** What are the roots of $h(X)$?

**b.** What are the possible symmetries of the roots of $h(X)$?

**c.** If you graph the roots of $h(X)$ in the complex plane, do their symmetries correspond to symmetries of the entire complex plane?

***d.*** The Galois group $G_h$ is the full set of symmetries.

**3.** Next consider the polynomial $p(X) = X^4 - 2$.

**a.** Find the roots of $p(X)$.

**b.** Show that not all symmetries are possible: for example if $\sqrt[4]{2}$ is mapped to $-\sqrt[4]{2}$, by some $\sigma \in G_p$ then $-\sqrt[4]{2}$ must be mapped to $\sqrt[4]{2}$ by $\sigma$.

***c.*** The symmetries of the roots of $p(X)$ turn out to be the symmetries of a square; in the identification, the vertices of the square cannot be arbitrarily labelled. Label the square in such a fashion that its symmetries correspond to those of $p(X)$.

**4.** Show that the following molecules and polynomials (or rather their roots) have identical symmetries:

**a.** $O_2$ and $X^2 + 1$,

**b.** CO and $X^2 - 1$,

**c.** $CO_2$ and $X^3 - 1$,

***d.*** $C_4H_8$ (cyclobutane) and $X^4 - 2$ (here treat each carbon atom and its two bonded hydrogens as a single entity)

****e.**** $C_4H_8$ (cyclobutane) and $(X^4 - 2)(X^4 + 2)$: the hydrogen atoms need to be treated in a very specific way to make this correspondence work!