NEW MEXICO MATHEMATICS CONTEST XXIX

NOVEMBER 11, 1995 FIRST ROUND THREE HOURS

1. Observe that the greatest common divisor of the pair {18, 24} is 6, and the least common multiple is 72. The same is true for the triple {12, 18, 24} and the quadruple {12, 18, 24, 36}.

   (a) Find all pairs of positive integers whose greatest common divisor is 95, and whose least common multiple is 1995.

   (b) Find all triples of positive integers whose greatest common divisor is 95, and whose least common multiple is 1995.

   (c) Find all quadruples of positive integers whose greatest common divisor is 95, and whose least common multiple is 1995.

2. (a) Of all the triangles having all 3 of their vertices at the vertices of a given cube, how many of them are right triangles ?

   (b) Describe the remaining triangles, if any, that are not right triangles. How many triangles are there of this type ?

3. Observe that

$$\begin{aligned} 3^2 + 4^2 &= 5^2, \\ 3^2 + 4^2 + 12^2 &= 13^2, \\ 3^2 + 4^2 + 12^2 + 84^2 &= 85^2. \end{aligned}$$

   Find a pair of positive integers x and y such that

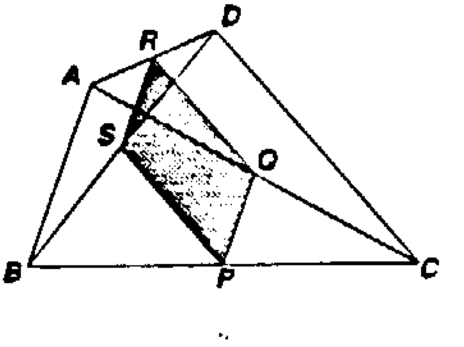
$$3^2 + 4^2 + 12^2 + 84^2 + x^2 = y^2.$$

   (It is *not* necessary to find *all* such pairs.)

4. In a quadrangle $ABCD$, suppose

$$\begin{aligned} \overline{AB} &= 5\,(cm), \\ \overline{CD} &= 8\,(cm), \\ \angle ABC &= 70^0, \\ \angle BCD &= 50^0, \end{aligned}$$

   and P, Q, R, S are the midpoints of BC, CA, AD, DB, respectively. Find the area of the parallelogram $PQRS$.



5. Suppose α and β are the two roots of the quadratic equation $3x^2+4x+5=0$. Find the value of

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}.$$

(Express your answer as a fraction in lowest terms.)

6. Suppose A and B are centers of the circles whose radii are 3 (cm) and 7 (cm), respectively, and $\overline{AB}=12\,(cm)$. Find the lengths of all common tangents PQ, where P and Q are on the circles A and B, respectively.

7. A function $g(x)$ is said to be *even*, while a function $h(x)$ is said to be *odd* if

$$g(-x)=g(x), \qquad h(-x)=-h(x) \qquad \text{for all } x.$$

For example, $g(x)=3+5x^2$ is even, while $h(x)=2x-x^3$ is odd.

(a) Given a function

$$f(x)=\frac{1}{1-x+x^2},$$

find a pair of functions $g(x)$ and $h(x)$, where $g(x)$ is even and $h(x)$ is odd such that

$$f(x)=g(x)+h(x) \quad \text{for all real numbers } x.$$

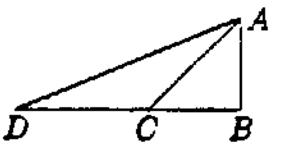
(b) Is such a decomposition of $f(x)$ unique ?

8. (a) Find integers a and b satisfying

$$\tan\frac{\pi}{8}=\sqrt{a}-b.$$

Hint : In the figure, $\angle ABC=\frac{\pi}{2}$, $\overline{AB}=\overline{BC}$, $\overline{AC}=\overline{CD}$.

$$\therefore\ \angle ADB=\frac{\pi}{8}, \quad \tan\frac{\pi}{8}=\overline{AB}/\overline{BD}.$$



(b) Find integers c and d satisfying

$$\tan\frac{\pi}{12}=c-\sqrt{d}.$$

(c) Find integers p, q, r, s satisfying

$$\tan\frac{\pi}{24}=(\sqrt{p}-\sqrt{q})(\sqrt{r}-s).$$