Knowing and Teaching
Elementary Mathematics

Teachers' Understanding of Fundamental
Mathematics in China and the United States

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Foreword

Lee S. Shulman
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This is a remarkable book. It is also remarkably easy to misunderstand its most important lessons. Liping Ma has conducted a study that compares mathematical understanding among U.S. and Chinese elementary school teachers as it relates to classroom teaching practices. What could be simpler? What could one possibly misconstrue? Let me count the ways.

- This book appears to be a comparative study of American and Chinese teachers of mathematics, but its most important contributions are not comparative, but theoretical.
- This book appears to be about understanding the content of mathematics, rather than its pedagogy, but its conception of content is profoundly pedagogical.
- This book appears to be about the practice of mathematics teaching, but it demands a hearing among those who set policy for teaching and teacher education.
- This book appears to be most relevant to the preservice preparation of teachers, but its most powerful findings may well relate to our understanding of teachers' work and their career-long professional development.
- This book focuses on the work of elementary school teachers, but its most important audience may well be college and university faculty members who teach mathematics to future teachers as well as future parents.

I shall try to clarify these somewhat cryptic observations in this foreword, but first a brief biographical note about Liping Ma.

Liping became an elementary school teacher courtesy of China's Cultural Revolution. An eighth-grade middle-school student in Shanghai, she was sent to "the countryside"—in her case a poor rural village in the mountainous area of South China—to be re-educated by the peasants working in the fields. After a few months, the village head asked Liping to become a teacher at the village school. As she has described it to me, she was a Shanghai teenager with but eight years of formal education struggling to teach all the subjects to two classes of kids in one classroom.
Over the next seven years, she taught all five grades and became principal of the school. A few years later, she would be hired as the Elementary School Superintendent for the entire county.

When she returned to Shanghai filled with curiosity about her new calling, she found a mentor in Professor Liu, who directed her reading of many of the classics of education—among them Confucius and Plato, Locke and Rousseau, Piaget, Vygotsky, and Bruner. Professor Liu eventually became president of East China Normal University where Liping earned a master's degree. She longed to study even more, and to pursue her further education in the United States. On the last day of 1988, she arrived in the United States to study at Michigan State University.

At Michigan State University, she worked with, among others, Sharon Feiman-Nemser and Suzanne Wilson in teacher education, with Deborah Ball and Magdalene Lampert in mathematics education, and with Lynn Paine in comparative education. She participated in the development and analysis of a national survey of elementary teachers' mathematical understandings, and marveled at the general misunderstandings that persisted among the U.S. teachers. They struck her as quite unlike the teachers she had come to know in China.

After a few years, her family chose to live in California, and Liping was admitted to the doctoral program at Stanford University to complete her coursework and dissertation. I served as her advisor and the Spencer Foundation awarded her a dissertation-year fellowship to complete the study that forms the basis for this book. This support, along with continued help from Michigan State, made possible her travel to China to collect the data from Chinese teachers. After completing her Ph.D., Liping was awarded a two-year postdoctoral fellowship to work with Alan Schoenfeld at Berkeley, where she continued her research and where her dissertation was transformed into this superb book.

What are the most important lessons to be learned from this book? Let us return to the list of misconceptions I presented earlier and discuss them more elaborately.

This book appears to be a comparative study of American and Chinese teachers of mathematics, but its most important contributions are not comparative, but theoretical. The investigation compares Chinese and American teachers and the Chinese, once again, know more. What could be simpler? But the key ideas of this book are not comparisons between American teachers and their Chinese counterparts. The heart of the book is Dr. Ma’s analysis of the kind of understanding that distinguishes the two groups. Chinese teachers are far more likely to have developed “profound understanding of fundamental mathematics.” To say that they “know more” or “understand more” is to make a deeply theoretical claim. They actually may have studied far less mathematics, but what they know they know more profoundly, more flexibly, more adaptively.

This book appears to be about understanding the content of mathematics, rather than its pedagogy, but its conception of content is profoundly pedagogical. Liping Ma set out to account for the differences in content knowledge and understanding between U.S. and Chinese elementary teachers, but her conception of understanding is critical. She has developed a conception of mathematical understanding that emphasizes those aspects of knowledge most likely to contribute to a teacher’s ability to explain important mathematical ideas to students. Thus, her stipulation of four properties of understanding—basic ideas, connectedness, multiple representations, and longitudinal coherence—offers a powerful framework for grasping the mathematical content necessary to understand and instruct the thinking of schoolchildren.

This book appears to be about the practice of mathematics teaching, but it demands a hearing among those who set policy for teaching and teacher education. Policymakers have become frantic in their insistence that future teachers demonstrate that they possess the knowledge of subject matter necessary to teach children. We are about to see tests of content knowledge for teachers proliferate among state licensing authorities. These cannot be tests that assess the wrong kind of knowledge. Liping Ma’s work should guide policymakers to commission the development of assessments that tap profound understanding of fundamental mathematics among future elementary teachers, not superficial knowledge of procedures and rules.

This book appears to be most relevant to the preservice preparation of teachers, but its most powerful findings may well relate to our understanding of teachers’ work and their career-long professional development. Liping was not satisfied to document the differences in understanding between Chinese and American teachers. She also inquired into the sources for those differences. A critical finding (echoed in the work on TIMMS by Stigler and Hiebert) is that Chinese teachers continue to learn mathematics and to refine their content understandings throughout their teaching careers. Teachers’ work in China includes time and support for serious deliberations and seminars on the content of their lessons. These are absolutely essential features of teacher work. American teachers are offered no opportunities within the school day for these collaborative deliberations, and therefore can teach for many years without deepening their understandings of the content they teach. Chinese teachers, in contrast, work in settings that create learning opportunities on a continuing basis.
This book focuses on the work of elementary school teachers, but its most important audience may well be college and university faculty members who teach mathematics to future teachers as well as future parents. Given our understanding of mathematical understanding for teaching, where can future teachers initially learn such mathematics? In China, they learn such mathematics from their own elementary and middle-school teachers, enhance that understanding in the content courses at their normal colleges (teacher-education colleges), and further sustain and develop their knowledge in practice. The only place to break the vicious cycle that limits the mathematical knowledge of U.S. teachers is in the development of far more effective mathematics courses in U.S. undergraduate programs. But current undergraduate mathematics programs seem to have no place for teaching fundamental mathematics for profound understanding. If anything, such knowledge is misconstrued as remedial instead of recognizing that it is rigorous and deserving of university-level instruction. Mathematics departments must take responsibility for serving this national priority for both future teachers and future citizens.

Although only now being published, copies of earlier drafts of this manuscript have been circulating in the mathematics community for some time. In a recent letter, Liping’s postdoctoral mentor, Professor Alan Schoenfeld of the University of California at Berkeley, described the response to prepublication copies of this book vividly.

Liping's manuscript has already gotten an amazing amount of notice. It’s an underground hit, perhaps the only manuscript I know that has the attention and favor of both sides of the “math wars.” Many world class mathematicians are rhapsodic about it; at the annual mathematics meetings, people like [he lists several leading professional mathematicians] were walking advertisements for the book. That's because it says content knowledge makes a difference. But at the same time, those who have reform perspectives—those who value a deep and connected view of mathematical thinking, and who understand that teacher competence includes having a rich knowledge base that contains a wide range of pedagogical content knowledge—find that the book offers riches regarding content, teacher preparation, and teacher professionalism.

This is indeed a valuable, enlightening book. It attests to the talent of its author, and to the Chinese and American learning environments that have nurtured that talent. It attests to the value of welcoming scholars from other nations to study in the United States. I urge all those who are seriously concerned about the quality of mathematics education in the United States to read this book, and to take its lessons seriously.

Acknowledgments

About 30 years ago, China was undergoing the so-called “Cultural Revolution.” Millions of students from cities were sent to rural areas. As one of them, I left Shanghai, where I was born and raised, and went to a small, poor village in a mountainous area of southern China. Seven of us teenagers—with seven or eight years of formal education—formed a “co-op” family there. We were supposed to make our living by working in the fields, and be re-educated by the peasants at the same time. A few months later, the head of the village came to me. He surprised me by asking me to become a teacher at the village elementary school—to educate their children. Most of the peasants in that mountain area were illiterate. They eagerly wanted to change this fate for their next generation.

Today, holding the book manuscript in my hands and looking back, I can clearly see the starting point of my career—that Shanghai girl who struggled very hard to teach all subjects to two grades of rural kids. It has been a long journey with laughter as well as tears. All the value of this book, if there is any, has been fostered throughout this journey.

This book would not have been possible without help along the way. The work occurred in two stages—the research and writing of my dissertation and its revision as a book. Both stages were accomplished with the assistance of numerous people and institutions.

First of all, I am grateful to the Spencer Foundation and the McDonnell Foundation for doctoral and postdoctoral fellowships that supported the writing and revision of my dissertation.

At Michigan State University in East Lansing—my “home town” in the United States—I was especially indebted to Professors Sharon Feiman-Nemser, Lynn Paine, and their families. My intellectual home was the TELT project and I benefited greatly from my interactions with project members. My work builds on TELT, both intellectually and through my use of questions developed by Deborah Ball and data collected by Deborah Ball, Sharon Feiman-Nemser, Perry Lanier, Michelle Parker, and Richard Prawat.

When I arrived in the United States, I only had thirty dollars in my pocket. Sharon, my advisor, worked hard to help me to find a research assistantship so that I could focus on my academic development. Her outstanding scholarship in teaching and research, as it does for all her other students, enlightened, and will continue to enlighten my research on teacher education.

Before I met Professor Lynn Paine, I could not tell from her phone call in Chinese that she was an American. She gave me my first and solid
upper stories, but it is the foundation that supports them and makes all
the stories (branches) cohere. The appearance and development of new
mathematics should not be regarded as a denial of fundamental mathe-
matics. In contrast, it should lead us to an ever better understanding of
elementary mathematics, of its powerful potentiality, as well as of the con-
ceptual seeds for the advanced branches.

PROFOUND UNDERSTANDING OF FUNDAMENTAL
MATHEMATICS

Indeed, it is the mathematical substance of elementary mathematics that
allows a coherent understanding of it. However, the understanding of
elementary mathematics is not always coherent. From a procedural per-
spective, arithmetic algorithms have little or no connection with other
topics, and are isolated from one another. Taking the four topics studied
as an example, subtraction with regrouping has nothing to do with mul-
digit multiplication, nor with division by fractions, nor with area and
perimeter of a rectangle.

Figure 5.1 illustrates a typical procedural understanding of the four
topics. The letters S, M, D, and G represent the four topics: subtraction
with regrouping, multidigit multiplication, division with fractions, and
the geometry topic (calculation of perimeter and area). The rectangles rep-
resent procedural knowledge of these topics. The ovals represent other pro-
cedural knowledge related to these topics. The trapezoids underneath the
rectangles represent pseudoconceptual understanding of each topic. The
dotted outlines represent missing items. Note that the understandings of
the different topics are not connected.

In Fig. 5.1 the four topics are essentially independent and few elements
are included in each knowledge package. Pseudoconceptual explanations
for algorithms are a feature of understanding that is only procedural. Some
teachers invented arbitrary explanations. Some simply verbalized the algo-
rithm. Yet even inventing or citing a pseudoconceptual explanation re-
quires familiarity with the algorithm. Teachers who could barely carry out
an algorithm tended not to be able to explain it or connect it with other
procedures, as seen in some responses to the division by fractions and
gometry topics. With isolated and underdeveloped knowledge packages,

the mathematical understanding of a teacher with a procedural perspective
is fragmentary.

From a conceptual perspective, however, the four topics are connected,
related by the mathematical concepts they share. For example, the concept
of place value underlies the algorithms for subtraction with regrouping
and multidigit multiplication. The concept of place value, then, becomes
a connection between the two topics. The concept of inverse operations
contributes to the rationale for subtraction with regrouping as well as to
the explanation of the meaning of division by fractions. Thus the concept
of inverse operations connects subtraction with regrouping and division
by fractions. Some concepts, such as the meaning of multiplication, are
shared by three of the four topics. Some, such as the three basic laws, are
shared by all four topics. Figure 5.2 illustrates how mathematical topics
are related from a conceptual perspective.

Although not all the concepts shared by the four topics are included,
Fig. 5.2 illustrates how relations among the four topics make them into a
network. Some items are not directly related to all four topics. However,
their diverse associations overlap and interface. The three basic laws ap-
peared in the Chinese teachers' discussions of all four topics.

In contrast to the procedural view of the four topics illustrated in Fig.
5.1, Fig. 5.3 illustrates a conceptual understanding of the four topics. The
four rectangles at the top of Fig. 5.3 represent the four topics. The ellipses
represent the knowledge pieces in the knowledge packages. White ellipses
represent procedural topics, light gray ones represent conceptual topics,

7Given a topic, a teacher tends to see other topics related to its learning. If it is procedural,
a teacher may see an explanation for it. If it is conceptual, a teacher may see a related
procedure or concept. This tendency initiates organization of a well-developed "knowledge
package." So I use the term "knowledge package" here for the group of topics that teachers
tend to see around the topic they are teaching.
Duckworth, a former student and colleague of Jean Piaget, believes we should keep learning of elementary mathematics and science “deep” and “complex” (1987, 1991). Inspired by Piaget’s concern for how far, instead of how fast, learning would go, she proposed the notion of “learning with depth and breadth” (1979). After a comparison between building a tower “with one brick on top of another” and “on a broad base or a deep foundation,” Duckworth said:

What is the intellectual equivalent of building in breadth and depth? I think it is a matter of making connections: breadth could be thought of as the widely different spheres of experience that can be related to one another; depth can be thought of as the many different kinds of connections that can be made among different facets of our experience. I am not sure whether or not intellectual breadth and depth can be separated from each other, except in talking about them. (p. 7)

I agree with Duckworth that intellectual breadth and depth “is a matter of making connections,” and that the two are interwoven. However, her definition of intellectual breadth and depth is too general for use in discussing mathematical learning. Moreover, she does not explain what their relationship is.

Based on my research, I define understanding a topic with depth as connecting it with more conceptually powerful ideas of the subject. The closer an idea is to the structure of the discipline, the more powerful it will be, consequently, the more topics it will be able to support. Understanding a topic with breadth, on the other hand, is to connect it with those of similar or less conceptual power. For example, consider the knowledge package for subtraction with regrouping. To connect subtraction with regrouping with the topics of addition with carrying, subtraction without regrouping, and addition without carrying is a matter of breadth. To connect it with concepts such as the rate of composing or decomposing a higher value unit or the concept that addition and subtraction are inverse operations is a matter of depth. Depth and breadth, however, depend on thoroughness—the capability to “pass through” all parts of the field—to weave them together. Indeed, it is this thoroughness which “glues” knowledge of mathematics into a coherent whole.

Both dimensions of the structure—basic principles and basic attitudes (Bruner, 1960/1977)—are very powerful in making connections. Unfortunately Fig. 5.3 is too simple to well illustrate the one-to-many relationships between general principles or attitudes and mathematical concepts or topics.

For educational researchers, the depth of teachers’ subject matter knowledge seems to be subtle and intriguing. On one hand, most would agree that teachers’ understanding should be deep (Ball, 1989; Grossman, Wilson, & Shulman, 1989; Marks, 1987; Steinberg, Marks, & Haymore, 1985; Wilson, 1988). On the other hand, because the term depth is “vague” and “elusive in its definition and measurements” (Ball, 1989; Wilson, 1988), progress in understanding it has been slow. Ball (1989) proposed three “specific criteria” for teachers’ substantive knowledge: correctness, meaning, and connectedness to avoid the term deep, which she considered a vague descriptor of teachers’ subject matter knowledge.
Of course, the reason that a profound understanding of elementary mathematics is possible is that first of all, elementary mathematics is a field of depth, breadth, and thoroughness. Teachers with this deep, vast, and thorough understanding do not invent connections between and among mathematical ideas, but reveal and represent them in terms of mathematics teaching and learning. Such teaching and learning tends to have the following four properties:

**Connectedness.** A teacher with PUFM has a general intention to make connections among mathematical concepts and procedures, from simple and superficial connections between individual pieces of knowledge to complicated and underlying connections among different mathematical operations and subdomains. When reflected in teaching, this intention will prevent students’ learning from being fragmented. Instead of learning isolated topics, students will learn a unified body of knowledge.

**Multiple Perspectives.** Those who have achieved PUFM appreciate different facets of an idea and various approaches to a solution, as well as their advantages and disadvantages. In addition, they are able to provide mathematical explanations of these various facets and approaches. In this way, teachers can lead their students to a flexible understanding of the discipline.

**Basic Ideas.** Teachers with PUFM display mathematical attitudes and are particularly aware of the "simple but powerful basic concepts and principles of mathematics" (e.g., the idea of an equation). They tend to revisit and reinforce these basic ideas. By focusing on these basic ideas, students are not merely encouraged to approach problems, but are guided to conduct real mathematical activity.

**Longitudinal Coherence.** Teachers with PUFM are not limited to the knowledge that should be taught in a certain grade; rather, they have achieved a fundamental understanding of the whole elementary mathematics curriculum. With PUFM, teachers are ready at any time to exploit an opportunity to review crucial concepts that students have studied previously. They also know what students are going to learn later, and take opportunities to lay the proper foundation for it.

These four properties are interrelated. While the first property, connectedness, is a general feature of the mathematics teaching of one with PUFM,

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10Kaput (1994) used this term to describe curricula, here I use it to describe the corresponding property for teacher knowledge. This property is related to an aspect of what Shulman (1986) called curricular knowledge.
Conclusion

As I said at the beginning of this book, the initial motivation for my study was to explore some possible causes of the unsatisfactory mathematics achievement of U.S. students in contrast to their counterparts in some Asian countries. In concluding, I would like to return to my original concern about the mathematics education of children in the United States. Having considered teachers' knowledge of school mathematics in depth, I suggest that to improve mathematics education for students, an important action that should be taken is improving the quality of their teachers' knowledge of school mathematics.

Although the intent of my study was not to evaluate U.S. and Chinese teachers' mathematical knowledge, it has revealed some important differences in their knowledge of school mathematics. It does not seem to be an accident that not one of a group of above average U.S. teachers displayed a profound understanding of elementary mathematics. In fact, the knowledge gap between the U.S. and Chinese teachers parallels the learning gap between U.S. and Chinese students revealed by other scholars (Stevenson et al., 1990; Stevenson & Stigler, 1992). Given that the parallel of the two gaps is not mere coincidence, it follows that while we want to work on improving students' mathematics education, we also need to improve their teachers' knowledge of school mathematics. As indicated in the introduction, the quality of teacher subject matter knowledge directly affects student learning—and it can be immediately addressed.

Teachers' subject matter knowledge develops in a cyclic process as depicted in Fig. 7.1.

Figure 7.1 illustrates three periods during which teachers' subject matter knowledge of school mathematics may be fostered. In China, the cycle spirals upward. When teachers are still students, they attain mathematical competence. During teacher education programs, their mathematical competence starts to be connected to a primary concern about teaching and learning school mathematics. Finally, during their teaching careers, as they empower students with mathematical competence, they develop a teacher's subject matter knowledge, which I call in its highest form PUFM.

Unfortunately, this is not the case in the United States. It seems that low-quality school mathematics education and low-quality teacher knowledge of school mathematics reinforce each other. Teachers who do not acquire mathematical competence during schooling are unlikely to have another opportunity to acquire it. The NCRTE (1991) study of teacher education programs indicates that most U.S. teacher preparation programs focus on how to teach mathematics rather than on the mathematics itself. After teacher preparation, teachers are expected to know how and what they will teach and not to require further study (Schifter, 1996a). This assumption is reflected in the U.S. educational structure: The National Commission on Teaching and America's Future (1997) found no system in place to ensure that teachers get access to the knowledge they need. This lack may be an important impediment to reform. In 1996, after two years of intensive study, the commission concluded that "Most schools and teachers cannot achieve the goals set forth in new educational standards, not because they are unwilling, but because they do not know how, and the systems they work in do not support them in doing so" (p. 1).
Reflecting on Chinese mathematics education one may notice that the upward spiral is not there by itself but is cultivated and supported by the solid substance of school mathematics in China. If the subject they taught did not have depth and breadth, how could Chinese teachers develop a profound understanding of it? In fact, there may exist another upward spiral in China—between substantial elementary mathematics and solid mathematics education. This contrasts with the sustained low levels in the United States, where inadequate elementary mathematics ("basic skills," "shopkeeper arithmetic") reinforces and is reinforced by unsatisfactory mathematics education. In the United States, it is widely accepted that elementary mathematics is "basic," superficial, and commonly understood. The data in this book explode this myth. Elementary mathematics is not superficial at all, and any one who teaches it has to study it hard in order to understand it in a comprehensive way.¹

How can these self-perpetuating relationships—between unsatisfactory student learning and inadequate teacher knowledge, between unsatisfactory mathematics education and inadequate elementary mathematics—be broken? How can the goals of reform be achieved? I conclude with some recommendations.

ADDRESS TEACHER KNOWLEDGE AND STUDENT LEARNING AT THE SAME TIME

First of all, I would like to indicate that although I take the gap in teacher knowledge as a factor in the gap in student learning, I do not regard improvement of teachers' knowledge as necessarily preceding improvement of students' learning. Rather, I believe both should be addressed simultaneously, and that work on each should support the improvement of the other. Because they are interdependent processes, we cannot expect to improve teachers' mathematical knowledge first, and in so doing automatically improve students' mathematics education.

As we saw in the previous chapter, a teacher's subject matter knowledge of school mathematics is a product of the interaction between mathematical competence and concern about teaching and learning mathematics. The quality of the interaction depends on the quality of each component. Given that their own schooling does not yet provide future teachers with sound mathematical competence, their base for developing solid teaching knowledge is weakened. As my data show, the group of Chinese ninth-grade students were more competent in elementary mathematics than the group of U.S. teachers, and in addition showed more conceptual understanding. This suggests that although Chinese teachers develop PUFM during their teaching careers, their schooling contributes a sound basis for it. Teacher candidates in the United States will not have this sound basis if student learning is not addressed.

The second reason that improving teachers' subject matter knowledge of mathematics cannot be isolated from improving school mathematics teaching is that, as I have revealed, the key period during which Chinese teachers develop a teacher's subject matter knowledge of school mathematics is when they teach it—given that they have the motivation to improve their teaching and the opportunity to do so. If this is true, it would be unrealistic to expect U.S. teachers' subject matter knowledge of school mathematics to be improved before mathematics education in school is improved. Improving teachers' subject matter knowledge and improving students' mathematics education are thus interwoven and interdependent processes that must occur simultaneously. What is needed, then, is a teaching context in which it is possible for teachers to improve their knowledge of school mathematics as they work to improve their teaching of mathematics.

ENHANCE THE INTERACTION BETWEEN TEACHERS' STUDY OF SCHOOL MATHEMATICS AND HOW TO TEACH IT

I have indicated that the key period during which Chinese teachers develop their deep understanding of school mathematics is when they are teaching it. However, this finding may not be true of teachers in the United States. The experienced U.S. teachers in this study did not perform better than their new colleagues in terms of subject matter knowledge. This finding agrees with that of the National Center for Research on Teacher Education (NCRTE, 1991). The question then is, Why did teaching mathematics in this country not produce PUFM among teachers?

I have observed that mathematics teaching in the United States lacks an interaction between study of the mathematics taught and study of how to teach it. Several factors hinder teachers from careful study of the school mathematics they teach. One is the assumption that I have already discussed—that elementary mathematics is "basic," superficial, and commonly understood.

Another assumption—that teachers do not need further study of the subject they teach—also hinders teachers' further study of school mathematics. Schifter wrote:

the notion that even experienced teachers can and should be expected to continue learning in their own classrooms contrasts sharply with the tradi-

¹Other scholars, such as Ball (1988d) have also revealed the falsity of the assumption that elementary mathematics is commonly understood.
tional assumption that becoming a teacher marks a sufficiency of learning. It is no great exaggeration to say that, according to the conventions of school culture, teachers, by definition, already know—know the content domain they are to teach, the sequence of lessons they must go through to teach it, and the techniques for imposing order on a roomful of students. (1996a, p. 163)

Even if teachers had the time and inclination for studying school mathematics, what would they study? Ball (1996) wrote, "it is not clear whether most curriculum developers write with teacher learning as a goal." H. Burkhardt (personal communication, May 11, 1998) said, "Professional developers, though they advocate a constructivist approach for kids, are only gradually allowing teachers to learn in a constructive fashion."

Textbook manuals offer teachers little guidance (Armstrong & Bezuk, 1995; Schmidt, 1996, p. 194), possibly because teachers are not expected to read them. Burkhardt (personal communication, May 11, 1998) said:

The math textbook provides a script (with stage directions) for the teacher to use in explaining the topic and guiding the lesson; the students are only expected to read and do the exercises at the end of the chapter. Nobody reads "teachers' guides" except on masters courses.

Although the results of the Third International Mathematics and Science Study indicate that elementary mathematics lessons in the United States tend to be based on the textbook (Schmidt, 1996, p. 104), little research focuses on exactly how teachers use textbooks (Freeman & Porter, 1989, pp. 67–88; Sosniak & Stodolsky, 1993). This research indicates that there may be wide variation in teachers' topic selection, content emphasis, and sequence of instruction. Textbooks are rarely followed from beginning to end (Schmidt, McKnight, & Raizen, 1997). Case studies suggest that teachers' knowledge plays a very important role in how textbook contents are selected and interpreted (Putnam, Heaton, Prawat, & Remillard, 1992). Even the teaching of one topic may have wide variation. As we have seen in the first three chapters of this book, different teachers may construe the same topic very differently.

In China, teaching a course is considered to be like acting in a play. Although an actor has to knows how to play very well and can interpret it in an original way, he or she is not supposed to write (or rewrite) the play. Indeed, a well-written play will not confine an actor's performance or creativity but will rather stimulate and inspire it.

The same can be true for teachers. Teaching can be a socially cooperative activity. We need good actors as well as good playwrights. A thoughtfully and carefully composed textbook carries wisdom about curriculum that teachers can "talk with" and that can inspire and enlighten them. In China, textbooks are considered to be not only for students, but also for teachers' learning of the mathematics they are teaching. Teachers study textbooks very carefully; they investigate them individually and in groups, they talk about what textbooks mean, they do the problems together, and they have conversations about them. Teacher's manuals provide information about content and pedagogy, student thinking and longitudinal coherence.

Time is an issue here. If teachers have to find out what to teach by themselves in their very limited time outside the classroom and decide how to teach it, then where is the time for them to study carefully what they are to teach? U.S. teachers have less working time outside the classroom than Chinese teachers (McKnight et al., 1987; Stigler & Stevenson, 1991), but they need to do much more in this limited time. What U.S. teachers are expected to accomplish, then, is impossible. It is clear that they do not have enough time and appropriate support to think through thoroughly what they are to teach. And without a clear idea of what to teach, how can one determine how to teach it thoughtfully?

**REFOCUS TEACHER PREPARATION**

I contend that teacher education is a strategically critical period during which change can be made. As the report of the Conference on the Mathematical Preparation of Elementary School Teachers points out:

It makes sense to attack the problems of elementary school mathematics education at the college level. All teachers go to college—it's where they expect to learn how to teach. Moreover, the task is almost manageable at the college level.... only about a thousand colleges educate teachers. (Cipra, 1992, p. 5)

Although my data do not show that Chinese teachers develop their PUFM during teacher preparation, this does not mean that the role of teacher preparation in improving teachers' knowledge of elementary mathematics should be minimized. On the contrary, in the vicious circle formed by low-quality mathematics education and low-quality teacher knowledge of school mathematics—a third party—teacher preparation may serve as the force to break the circle.

Refocusing teacher preparation, however, creates another important task for educational research—rebuiding a solid and substantial school mathematics for teachers and students to learn. What we should do is to rebuild a substantial school mathematics with a more comprehensive understanding of the relationship between fundamental mathematics and new advanced branches of the discipline. To rebuild a substantial school mathematics for today is a task for mathematics education researchers. Indeed, unless such a school mathematics is developed, the mutual reinforcement of low-level content and teaching will not be undone.
UNSTAND THE ROLE THAT CURRICULAR MATERIALS, INCLUDING TEXTBOOKS, MIGHT PLAY IN REFORM

Like textbooks, reform documents such the California Framework (1985) and the National Council of Teachers of Mathematics (NCTM) Standards (1989) lend themselves to multiple interpretations (Putnam et al., 1992) that depend on the reader's knowledge and beliefs about mathematics, teaching, and learning.

The Professional Standards for Teaching Mathematics (NCTM, 1991, p. 32) says that "textbooks can be useful resources for teachers, but teachers must also be free to adapt or depart from texts if students' ideas and conjectures are to help shape teachers' navigation of the content." Ferrucci (1997) pointed out that discontinuing the use of textbooks may be viewed as being consistent with this statement. Others characterize reform teachers as "using the textbook as a supplement to the curriculum" for homework, practice, and review; in contrast, traditional teachers depend on the text to guide the scope and sequence of the curriculum (Kroll & Black, 1993, p. 431).

Because of dissatisfaction with textbooks (Ball, 1993b; Heaton, 1992; Schifter, 1996b) or because they were encouraged to do so in preservice programs (Ball & Feiman-Nemser, 1988), some reform-minded teachers independently organize their own curricula, make their own materials, and implement the lessons they have designed (Heaton, 1992; Shimahara & Sakai, 1995; Stigler, Fernandez, & Yoshida, 1996, p. 216; for narratives from SummerMath teachers, see Schifter, 1996c, 1996d). Ball and Cohen (1996) wrote:

educators often disparage textbooks, and many reform-oriented teachers repudiate them, announcing disdainfully that they do not use texts. This idealization of professional autonomy leads to the view that good teachers do not follow textbooks, but instead make their own curriculum.... This hostility to texts, and the idealized image of the individual professional, have inhibited careful consideration of the constructive role that curriculum might play. (p. 6)

Teachers need not have an antagonistic relationship with textbooks. My data illustrate how teachers can both use and go beyond the textbook. For example, Chinese teachers' knowledge packages are consistent with the national curriculum. But the student's idea that Mr. Mao "caught" (chapter 6) and the nonstandard methods of subtraction with regrouping, multidigit multiplication, and division by fractions described by the Chinese teachers were not in the textbook.

Teacher's manuals can explain curriculum developers' intentions and reasons for the way topics are selected and sequenced. Manuals can also provide very specific information about the nature of students' responses to particular activities (Magidson, 1994 April; Stigler, Fernandez, & Yoshida, 1996). Information about student responses can support teachers who focus on student thinking. However, such information may be useless if teachers do not recognize its significance or do not have time and energy for careful study of manuals (Magidson, 1994 April).

UNSTAND THE KEY TO REFORM: WHATEVER THE FORM OF CLASSROOM INTERACTIONS MIGHT BE, THEY MUST FOCUS ON SUBSTANTIVE MATHEMATICS

Like the use of textbooks, the kind of teaching advocated by reform documents is subject to different interpretations. For example, Putnam and his colleagues (1992) interviewed California teachers and state and district mathematics educators. Some thought the primary focus of the 1985 California Framework was what to teach—"important mathematical content"; others thought it was how to teach—"a call to use manipulatives and cooperative groups" (p. 214). During 1992 and 1993, the Recognizing and Recording Reform in Mathematics Education Project studied schools across the United States. Project members Ferrini-Mundy and Johnson (1994) noted that superficial efforts can pass for change. "Mathematics classrooms can appear to be quite Standards-oriented, with calculators in evidence, students working in groups, manipulatives available, and interesting problems under discussion" (p. 191), but investigators need a deeper understanding of what is happening in these classrooms.

This dichotomy sharpens when we consider Chinese teachers' classrooms. On one hand, mathematics teaching in Chinese classrooms, even by a teacher with PUFM, seems very "traditional"; that is, contrary to what is advocated by reform. Mathematics teaching in China is clearly textbook based. In Chinese classrooms, students sit in rows facing the teacher, who is obviously the leader and maker of the agenda and direction in classroom learning. On the other hand, one can see in Chinese classrooms, particularly in those of teachers with PUFM, features advocated by reform—teaching for conceptual understanding, students' enthusiasm and opportunities to express their ideas, and their participation and contribution to their own learning processes. How can these seemingly contradictory features—some protested against and some advocated by reform—occur at the same time? What might this intriguing contrast imply for reform efforts in the United States?

The perspective of Cobb and his colleagues (Cobb, Wood, Yackel, & McNeal, 1992) helps to explain this puzzle. Cobb and his associates view the essence of the current reform as a change of classroom mathematics tradition and contend that traditional and reform instruction differ in "the
quality of the taken-to-be-shared or normative meanings and practices of mathematics" rather than in "rhetorical characterizations."

In their case study of two classrooms, one with "a tradition of school mathematics" where knowledge was "transmitted" from the teacher to "passive students" and one with "a tradition of inquiry mathematics" in which "mathematical learning was viewed as an interactive, constructive, problem-centered process," the scholars found that in both the teachers and the students actively contributed to the development of their classroom mathematics tradition, while in both classrooms the teachers expressed their "institutionalized authority" during the process. Cobb and his associates suggest that "meaningful learning" may be mere rhetoric in mathematics education because "the activity of following procedural instructions can be meaningful for students" in certain classroom mathematics traditions. The transmission metaphor that describes traditional mathematics teaching as the attempt to transmit knowledge from the teacher to passive students may be appropriate only "in the political context of reform" (p. 34).

In this sense, although the mathematics teaching in Chinese teachers' classrooms does not meet some "rhetorical characterizations" of the reform, it is actually in the classroom mathematics tradition advocated by the current reform. In fact, even though the classroom of a Chinese teacher with PUFM may look very "traditional" in its form, it transcends the form in many aspects. It is textbook based, but not confined to textbooks. The teacher is the leader, but students' ideas and initiatives are highly encouraged and valued.

On the other hand, from a teacher who cannot provide a mathematical explanation of algorithms for subtraction with regrouping, multidigit multiplication, or division by fractions; from a teacher who cannot provide a correct representation for the meaning of an arithmetical operation such as division by fractions; or from a teacher who is not motivated to explore new mathematical claims, what kind of "teaching for understanding" can we expect?

To make the point more clearly we can think about a classroom like that of Ball (1993a, 1993b, 1996), considered by some to be a model of current reform:

In the classroom centered on student thinking and discussion—the classroom envisioned by mathematics education reformers—the children regularly disperse into small groups where they work together on problems, while the teacher visits around the classroom listening for significant mathematical issues and considering what types of intervention, if any, are appropriate. And when the children reassemble to compare their ideas and solutions, her questions facilitate discussion. (Schifter, 1996b, p. 3)

That is not at all the way the Chinese classrooms are organized. What I want to point out is, however, that even though they look very different, the difference is superficial. If you look carefully at the kind of mathematics that the Chinese students are doing and the kind of thinking they have been encouraged to engage in, and the way in which the teachers' interactions with them foster that kind of mental and mathematical process, the two kinds of classrooms are actually much more similar than they appear. On the other hand, although the fact that so many U.S. elementary teachers have children in groups facing each other and using manipulatives may mean that their classrooms look more like Ball's classroom when you walk in, nevertheless, neither the mathematics nor the mathematical thinking that the students are doing nor what the teacher is attempting to help them understand are the same. The real mathematical thinking going on in a classroom, in fact, depends heavily on the teacher's understanding of mathematics.

Another point I would like to make is that the change of a classroom mathematics tradition may not be a "revolution" that simply throws out the old and adopts the new. Rather, it may be a process in which some new features develop out of the old tradition. In other words, the two traditions may not be absolutely antagonistic to each other. Rather, the new tradition embraces the old—just as a new paradigm in scientific research does not completely exclude an old one but includes it as a special case.

In real classroom teaching, the two traditions may not be distinguished from each other clearly, or they may not be so "pure" as has been described. For example, my study indicates that teachers with PUFM never ignore the role of "procedural learning" no matter how much they emphasize "conceptual understanding."

Moreover, this research suggests that teachers' subject matter knowledge of mathematics may contribute to a classroom mathematics tradition and its alteration. A "taken-to-be-shared mathematical understanding" that marks a classroom tradition cannot be independent from the mathematical knowledge of people in the classroom, especially that of the teacher who is in charge of the teaching process. If a teacher's own knowledge of the mathematics taught in elementary school is limited to procedures, how could we expect his or her classroom to have a tradition of inquiry mathematics? The change that we are expecting can occur only if we work on changing teachers' knowledge of mathematics.

I would like to end with a quotation from Dewey (1902/1975):

But here comes the effort of thought. It is easier to see the conditions in their separateness, to insist upon one at the expense of the other, to make antagonists of them, than to discover a reality to which each belongs. (p. 91)