1. Let $f \in C([0, 1])$. Show that \( \lim_{n \to \infty} \int_0^1 x^n f(x) dx = 0 \).

2. Let \( |a| < 1 \) and \( f_n(t) = \sin \left( (2n + 1)\frac{\pi}{2} t \right), \ t \in \mathbb{R} \).

   (a) Show that for any \( a \) and \( t \) as above, the series \( \sum_{n=1}^{\infty} a^{2n+1} f_n(t) \) converges absolutely.

   (b) Determine if \( f(t) = \sum_{n=1}^{\infty} a^{2n+1} f_n(t) \) is a continuous function of \( t \).

   (c) Show that \( \frac{1}{\pi} \ln \frac{1+a}{1-a} - \frac{2}{\pi} a = \int_0^1 f(t) dt \).

3. (Lebesgue covering theorem) Let \((K, d)\) be a compact metric space and \( \mathcal{U} \) an open cover of \( K \). Show that there is an \( \epsilon > 0 \) such that \( \{B(x, \epsilon)\}_{x \in K} \) is a refinement of \( \mathcal{U} \), i.e., every \( B(x, \epsilon) \) is contained in some open set from \( \mathcal{U} \).

4. Let \((X, d)\) be a compact metric space and \( F \) a mapping \( F : X \to X \) such that \( d(F(x), F(y)) < d(x, y) \) for all \( x, y \in X \).

   (a) Show that the function \( g : X \to [0, \infty) \), defined by \( g(x) = d(x, F(x)) \), is a continuous function, whose minimum must be zero.

   (b) Show that the equation \( F(x) = x \) has exactly one solution, i.e., \( F \) has exactly one fixed point.
5. (a) State the inverse function theorem on $\mathbb{R}^n$.

(b) Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map such that its Jacobian is non-zero at every point and $F$ is proper (i.e. the pre-image of any compact set is a compact set). Show that $F$ is onto. Consider the exponential function $f(x) = e^x$, $f : \mathbb{R} \to \mathbb{R}$, is there a contradiction with the previous statement? Explain.

6. An electric charge $q$ located at the origin produces the electric field

$$
\vec{E} = \frac{q\vec{R}}{4\pi\epsilon\|\vec{R}\|^3},
$$

where $\vec{R} = xi + yj + zk$ and $\epsilon$ is a physical constant, called the electric permittivity.

(a) Show that

$$
\int \int_S \vec{E} \cdot \vec{N} dS = 0
$$

if the closed surface $S$ does not enclose the origin (we are assuming $S$ to be a piecewise smooth surface bounding a bounded domain in $\mathbb{R}^3$, and $\vec{N}$ is the outer normal to the surface $S$).

(b) Show that

$$
\int \int_S \vec{E} \cdot \vec{N} dS = \frac{q}{\epsilon}
$$

if the closed surface $S$ encloses the origin.

7. Let $\vec{F} = yi - xj + zk$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$

(a) along the segment $C$ joining $(1, 0, 0)$ to $(1, 0, 4)$;

(b) along the helix $C$ given by $x = \cos t$, $y = \sin t$, $z = \frac{4t}{\pi}$, for $0 \leq t \leq 2\pi$.

(c) Decide whether $\vec{F}$ is a conservative vector field or not.