

**Real Analysis Qualifying Exam**  
**January 13, 2003**

**Instructions:** There are eight problems, please do all of them. Start each problem on a new sheet of paper and write on one side of each sheet of paper. Remember to write your Social Security number on all pages and to number them. Good luck!!

**Problem 1:**

Let  $E$  be an infinite subset of a compact set  $K$  (every open cover has a finite subcover) in a metric space  $X$ . Define metric space and limit point and show that  $E$  has a limit point in  $K$ .

**Problem 2:** Let  $\{p_n\}$  be a Cauchy sequence in  $\mathbf{R}$ . Show that  $\{p_n\}$  converges in  $\mathbf{R}$ . *use compactness OK*

**Problem 3:** Let  $f : X \rightarrow Y$  where  $X$  and  $Y$  are metric spaces.

- (a) Give the " $\epsilon - \delta$ " definition of continuity.
- (b) Suppose  $f^{-1}(V)$  is open in  $X$  for every open  $V$  in  $Y$ . Show  $f$  is continuous on  $X$ .

**Problem 4:**

Let  $\phi, \psi : \mathbf{R} \rightarrow \mathbf{R}$  be twice differentiable functions and let  $a \in \mathbf{R}$ . Define  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  by  $f(x, y) = \phi(x - ay) + \psi(x + ay)$  for all  $(x, y) \in \mathbf{R}^2$ . Show that  $\frac{\partial^2 f}{\partial y^2} - a^2 \frac{\partial^2 f}{\partial x^2}$  is a constant. Find the value of the constant.

**Problem 5:** Let  $a, b \in \mathbf{R}$ ,  $a < b$ , and let  $f$  be a bounded real-valued function on  $[a, b]$ .

- (a) Define the Riemann integral of  $f$  between  $a$  and  $b$  in terms of the upper and lower sums,  $U(P, f)$  and  $L(P, f)$ . Also define  $P$ ,  $U$  and  $L$ .
- (b) Show that  $f$  is Riemann integrable on  $[a, b]$  if and only if, for every  $\epsilon > 0$  there exists a partition  $P$  such that  $U(P, f) - L(P, f) < \epsilon$ .

**Problem 6:**

Prove the following.

Let  $f$  be Riemann integrable on  $[a, b]$ . For  $a \leq x \leq b$ , put  $F(x) = \int_a^x f(t)dt$ . Then  $F$  is continuous on  $[a, b]$ ; furthermore, if  $f$  is continuous at a point  $x_0$  of  $[a, b]$ , then  $F$  is differentiable at  $x_0$ , and  $F'(x_0) = f(x_0)$ .

**Problem 7:**

Prove the following.

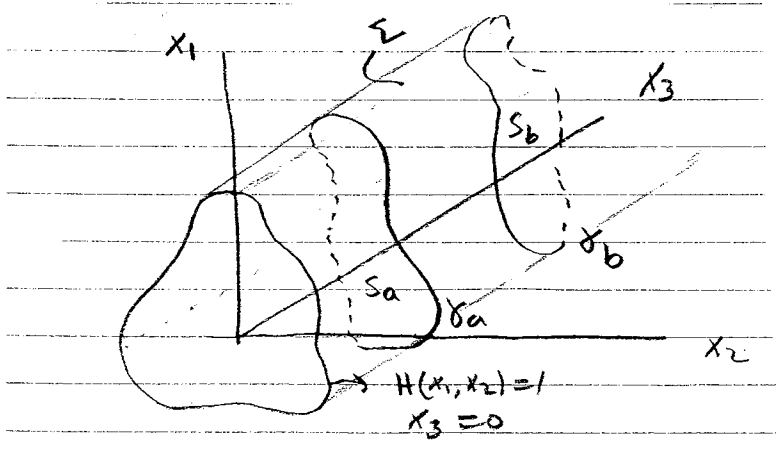
If  $\{f_n\}$  is a sequence of continuous functions on a set  $E$  in a metric space, and if  $f_n \rightarrow f$  uniformly on  $E$ , then  $f$  is continuous on  $E$ . As part of your argument give the precise definition of uniform convergence.

**Problem 8:**

(a) The divergence theorem for a vector field  $\mathbf{A}$  on  $\mathbf{R}^3$  can be written  $\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot \mathbf{n} dS$ . Define  $V, S, \nabla \cdot \mathbf{A}$  and  $\mathbf{n}$ . Let  $\mathbf{A} = \nabla \times \mathbf{B}$  and show that  $\int_S \mathbf{A} \cdot \mathbf{n} dS = 0$ .

(b) Stokes theorem for a vector field  $\mathbf{B}$  on  $\mathbf{R}^3$  can be written  $\int_S (\nabla \times \mathbf{B}) \cdot \mathbf{n} dS = \oint B_1 dx_1 + B_2 dx_2 + B_3 dx_3$ . Define  $S, \mathbf{n}$  and the line integral as a Riemann integral.

(c) Let  $H(x_1, x_2) = 1$  define a simple closed curve in the  $(x_1, x_2)$  plane and consider the tube in  $\mathbf{R}^3$  with surface  $T = \{(x_1, x_2, x_3) : x_3 \in \mathbf{R}, H(x_1, x_2) = 1\}$ . Take a piece of the tube defined by two ends with surfaces  $S_a$  and  $S_b$  and let  $\Sigma$  be the surface of the tube ( $H = 1$ ) between  $S_a$  and  $S_b$ . Let  $\gamma_a$  and  $\gamma_b$  be the boundaries of  $S_a$  and  $S_b$ .



Note  
 $S_a$  and  $S_b$  are not necessarily planar figures.

Show that  $\int_{\gamma_a} x_1 dx_2 - H(x_1, x_2) dx_3 = \pm \int_{\gamma_b} x_1 dx_2 - H(x_1, x_2) dx_3$ . Choose orientations which give a + sign and indicate your orientation on the figure.

Hint: Let  $\mathbf{A} = \nabla \times \mathbf{B}$  with  $\mathbf{B} = (0, x_1, -H(x_1, x_2))$ .