Real Analysis, Spring 2005, Qualifying Exam

Instructions: Complete all problems. Start each problem on a new page, number the pages, and put only your Social Security number on each page. Clear and concise answers with good justification will improve your score.

1. Let \( K \) be a compact metric space with metric \( d \) and let \( f \) be a continuous real-valued function defined on \( K \) (i.e. \( f \in C(K) \)). Prove that the graph of the function \( f \)

\[
\Gamma_f = \{(x, y) : x \in K, y = f(x)\}
\]

is a compact set in the metric space \((K \times \mathbb{R}, \rho)\), where

\[
\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + |y_1 - y_2|.
\]

2. Let \( f, g \) be real-valued continuous functions defined on the interval \([0, 1]\), i.e. \( f, g \in C[0, 1] \). Consider the uniform metric on \( C[0, 1] \) given by

\[
\rho(f, g) := \sup_{t \in [0,1]} |f(t) - g(t)|.
\]

For \( f \in C[0, 1] \), define \( F(f) \) as the continuous function defined for each \( t \in [0, 1] \) by

\[
F(f)(t) = \int_0^t u f(u)du.
\]

Show that \( F : C[0, 1] \rightarrow C[0, 1] \) is a contraction, i.e.

\[
\rho(F(f), F(g)) \leq \alpha \rho(f, g), \quad f, g \in C[0, 1]
\]

with some \( \alpha \in (0, 1) \). Explain why this implies that the equation

\[
f(t) = \int_0^t u f(u)du, \quad t \in [0, 1]
\]

has a unique solution \( f \in C[0, 1] \).

3. Let \( f \in C[0, 1] \) and suppose \( f(t) > 0 \) for all \( t \in [0, 1] \). Define \( \theta_n > 0 \) by the following equation:

\[
\int_0^{\theta_n} f(x)dx = \frac{1}{n} \int_0^1 f(x)dx.
\]

Find the following limit

\[
\lim_{n \to \infty} n \theta_n.
\]
4. Suppose that \( f \) is differentiable in the closed interval \([a, b]\) and that its second derivative \( f'' \) exists in the open interval \((a, b)\). Suppose also that
\[
f(a) = f(b), \quad f'(a) = f'(b) = 0.
\]
Show that there exist two points \( c_1, c_2 \in (a, b), c_1 \neq c_2 \) such that
\[
f''(c_1) = f''(c_2).
\]

5. Consider the following series
\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots
\]
In other words, the general term is \(-\frac{1}{n}\) if \( n = 2^k \) for some \( k = 1, 2, \ldots \) and its equal to \( \frac{1}{n} \) otherwise. Prove that the series diverges.

6. Prove that in some neighborhood of \((0, 0) \in \mathbb{R}^2\) there exists unique continuously differentiable function \( f \) such that in this neighborhood
\[
x_1 + x_2 + f(x_1, x_2) - \sin(x_1x_2f(x_1, x_2)) = 0.
\]
Find the partial derivatives of the function \( f \) at \((0, 0)\).

Please state carefully any theorem that you use in this exercise and the next.

7. Let \( E \subset \mathbb{R}^3 \) be open, suppose \( u \) and \( v \) are twice continuous differentiable real-valued functions on \( E \), i.e. \( u, v \in C^2(E) \). Let \( \nabla v \) denote the gradient of \( v \), \( \nabla^2 v = \nabla \cdot (\nabla v) = \sum_{i=1}^{3} \frac{\partial^2 v}{\partial x_i^2} \) denote the Laplacian of \( v \).

Assume \( \Omega \) is a closed subset of \( E \) with a positively oriented boundary \( \partial \Omega \), and let \( \mathbf{n} \) denote the outward normal to \( \partial \Omega \).

Prove Green’s identities,
\[
\int_{\Omega} [u \nabla^2 v + (\nabla u) \cdot (\nabla v)] \, dV = \int_{\partial \Omega} (u \nabla v) \cdot \mathbf{n} \, dA,
\]
and
\[
\int_{\Omega} [u \nabla^2 v - v \nabla^2 u] \, dV = \int_{\partial \Omega} (u \nabla v - v \nabla u) \cdot \mathbf{n} \, dA.
\]