PDE Exam, Winter 1999

1) Let $f(r)$ denote a smooth function defined for all real $r > 0$ and set

$$u(x,y) = f(\sqrt{x^2+y^2}), \quad (x,y) \neq (0,0) \ .$$

a) Show that

$$\Delta u = f'' + \frac{1}{r} f'$$

where $\Delta u = u_{xx} + u_{yy}$.
b) Use a) to determine all functions $u(x,y)$ which depend only on $r = \sqrt{x^2+y^2}$ and which satisfy $\Delta u = 0$ in the whole plane except at $(x,y) = (0,0)$.

2) Consider the nonlinear equation

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

with initial condition

$$u(x,0) = \sin x \ .$$

a) Explain how characteristics can be used to construct a smooth solution in some time interval $0 \leq t \leq t_0, t_0 > 0$.
b) Explain why a smooth solution does not exist beyond $t = 1$.
c) Consider

$$v_t + vv_x = -v, \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

with the same initial condition,

$$v(x,0) = \sin x \ .$$

Can one construct a smooth solution for all $t \geq 0$?

3) Consider the equations

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

and

$$v_t = -v_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

with initial condition

$$u(x,0) = v(x,0) = \sin(nx)$$

where n is an integer.

a) Determine solutions, $u(x,t)$ and $v(x,t)$, which are 2π–periodic in x.
b) Use the two examples to discuss the concepts of well–posedness and ill–posedness.