1. Let \( A \in \mathbb{R}^{n \times n} \) be an invertible, symmetric positive definite matrix, \( b \in \mathbb{R}^n \). This problem regards the method of steepest descent to find the solution \( x^* \) of \( Ax = b \). Steepest descent is an iterative method that defines a sequence \( x_n \) which converges to the minimizer of the function

\[
\phi(x) = \frac{1}{2} x^T A x - x^T b.
\]

(a) Let \( e(x) = x - x^* \) and \( ||x||_A = \sqrt{x^T A x} \), where \( x^* = A^{-1} b \) solves \( Ax = b \). Prove that \( x \) minimizes \( \phi(x) \) if and only if \( x \) minimizes \( ||e(x)||_A \), and thus \( x = x^* \), unique.

(b) Derive a formula for \(-\nabla \phi\).

(c) The vector \(-\nabla \phi(x)\) points in the direction of steepest descent of \( \phi \) at \( x \). The method of steepest descent consists of iterating

\[
x_{n+1} = x_n - \alpha_n \nabla \phi(x_n)
\]

starting from an initial guess \( x_0 \). That is, one steps from \( x_n \) to \( x_{n+1} \) by moving along the direction of steepest descent. Determine the optimal step length \( \alpha_n \) that minimizes \( \phi(x_{n+1}) \). Explain why the method always converges to the minimizer \( x^* \) of \( \phi \).

(d) Write down an algorithm for the full steepest descent iteration. There are three operations inside the main loop.

2. Consider a matrix \( A \in \mathbb{C}^{n \times n} \), vector \( x \in \mathbb{C}^n \) with \( x \neq 0 \). The Rayleigh Quotient of \( A \), \( R_A(x) \), is defined as the quantity

\[
R_A(x) = \frac{x^* A x}{x^* x}.
\]

(a) State the definition of a **Hermitian** matrix and characterize its spectrum, eigenvectors and diagonalizability.

(b) Prove that if \( A \) is Hermitian then \( R_A(x) \) is real and

\[
\lambda_1 \leq R_A(x) \leq \lambda_n,
\]

where \( \lambda_1 \) (resp. \( \lambda_n \)) is the least (resp. greatest) eigenvalue of \( A \).

(c) Determine the maximum and minimum values of the ratio

\[
R(x) = \frac{x_1^2 - 2x_1x_2 + 2x_2^2 + x_3^2}{x_1^2 + x_2^2 + 4x_3^2}
\]

by identifying with the Rayleigh quotient of an appropriate Hermitian matrix.
3. (Proof and application of Gershgorin’s theorem.) Let \( A \in \mathbb{R}^{n \times n} \).

(a) Prove: Every eigenvalue of \( A \) lies in at least one of the \( n \) circular disks in the complex plane with centers \( a_{ii} \) and radii \( \sum_{j \neq i}^{n} |a_{ij}| \),

\[
|\lambda - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}|
\]

(Hint: let \( \lambda \) be an eigenvalue of \( A \), and \( x \) be a corresponding eigenvector with largest entry 1.)

(b) Prove: Moreover, if \( m \) of these disks form a connected domain that is disjoint from the other \( n - m \) disks, then there are precisely \( m \) eigenvalues of \( A \) within this domain. (Hint: Consider the continuous deformation \( C(t) = D + tB \) of \( A \), where \( D = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \) and \( B = A - D \), and use the fact that the eigenvalues of a matrix are continuous functions of its entries.)

(c) Explain why in (a) the deleted absolute row sums can be replaced by the deleted absolute column sums

\[
|\lambda - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ji}|
\]

(d) Give estimates based on Gershgorin’s theorem ((a,b) above) for the eigenvalues of

\[
A = (8 1 0; 1 4 \epsilon; 0 1), \quad |\epsilon| < 1
\]

(e) Find a way to establish the tighter bound \( |\lambda_3 - 1| \leq \epsilon^2 \) on the smallest eigenvalue of \( A \). (Hint: consider a diagonal similarity transformation)

(f) Explain why Gershgorin’s theorem implies that \( A = (1 0 -2 0; 0 1 2 0; -4; 1 0 -1 0; 0 5 0 0) \) has at least 2 real roots.

4. Consider a matrix \( A \in \mathbb{C}^{n \times n} \).

(a) Define the spectral radius, \( \rho(A) \) and the matrix norms \( ||A||_k \), \( k = 1, 2, \infty \).

(b) Show that \( \rho(A) \leq ||A|| \) where ||...|| is any matrix norm induced by a vector norm.

(c) Show that

\[
||A||_2^2 = \rho(A^*A) = \sigma_1^2
\]

where \( \sigma_1 \) is the largest singular value of \( A \).

(d) Show that

\[
||A||_2^2 \leq ||A||_1 ||A||_\infty .
\]