MS/PhD QUALIFYING EXAMINATION

Numerical Analysis, Fall 2007

1. Let $A \in \Re^{n\times n}$ be an invertible, symmetric positive definite matrix, $\mathbf{b} \in \Re^n$. This problem regards the method of steepest descent to find the solution $\mathbf{x}^*$ of $A\mathbf{x} = \mathbf{b}$. Steepest descent is an iterative method that defines a sequence $\mathbf{x}_n$ which converges to the minimizer of the function

$$\phi(x) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b} \ .$$

   (a) Let $e(\mathbf{x}) = \mathbf{x} - \mathbf{x}^*$ and $||x||_A = \sqrt{x^T A\mathbf{x}}$. where $\mathbf{x}^* = A^{-1}\mathbf{b}$ solves $A\mathbf{x} = \mathbf{b}$. Prove that $\mathbf{x}$ minimizes $\phi(\mathbf{x})$ if and only if $\mathbf{x}$ minimizes $||e(\mathbf{x})||_A$, and thus $\mathbf{x} = \mathbf{x}^*$, unique.

   (b) Derive a formula for $-\nabla\phi$.

   (c) The vector $-\nabla\phi(\mathbf{x})$ points in the direction of steepest descent of ϕ at $\mathbf{x}$. The method of steepest descent consists of iterating

$$x_{n+1} = x_n - \alpha_n \nabla\phi(x_n)$$

   starting from an initial guess $\mathbf{x}_0$. That is, one steps from x_n to x_{n+1} by moving along the direction of steepest descent. Determine the optimal step length α_n that minimizes $\phi(\mathbf{x}_{n+1})$. Explain why the method always converges to the minimizer $\mathbf{x}^*$ of ϕ.

   (d) Write down an algorithm for the full steepest descent iteration. There are three operations inside the main loop.

2. Consider a matrix $A \in \mathbf{C}^{n\times n}$, vector $\mathbf{x} \in \mathbf{C}^n$ with $\mathbf{x} \neq 0$. The Rayleigh Quotient of A, $\mathcal{R}_A(\mathbf{x})$, is defined as the quantity

$$\mathcal{R}_A(\mathbf{x}) = \frac{x^* A x}{x^* x} \ .$$

   (a) State the definition of a **Hermitian** matrix and characterize its spectrum, eigenvectors and diagonalizability.

   (b) Prove that if A is Hermitian then $\mathcal{R}_A(\mathbf{x})$ is real and

$$\lambda_1 \leq \mathcal{R}_A(\mathbf{x}) \leq \lambda_n \ ,$$

   where λ_1 (resp. λ_n) is the least (resp. greatest) eigenvalue of A.

   (c) Determine the maximum and minimum values of the ratio

$$R(\mathbf{x}) = \frac{x_1^2 - 2x_1x_2 + 2x_2^2 + x_3^2}{x_1^2 + x_2^2 + 4x_3^2}$$

   by identifying with the Rayleigh quotient of an appropriate Hermitian matrix.

3. (Proof and application of Gershgorin's theorem.) Let $A \in \Re^{n\times n}$.

   (a) Prove: Every eigenvalue of A lies in at least one of the n circular disks in the complex plane with centers a_{ii} and radii $\sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}|$,

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}|$$

   (Hint: let λ be an eigenvalue of A, and $\mathbf{x}$ be ta corresponding eigenvector with largest entry 1.)

   (b) Prove: Moreover, if m of these disks form a connected domain that is disjoint from the other $n-m$ disks, then there are precisely m eigenvalues of A within this domain. (Hint: Consider the continuous deformation $C(t) = D + tB$ of A, where $D = \mathrm{diag(a_{11}, a_{22}, \ldots, a_{nn})}$ and $B = A - D$, and use the fact that the eigenvalues of a matrix are continuous functions of its entries.)

   (c) Explain why in (a) the deleted absolute row sums can be replaced by the deleted absolute column sums

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ji}|$$

   (d) Give estimates based on Gershgorin's theorem ((a,b) above) for the eigenvalues of

$$A = (8\ 1\ 0; 1\ 4\ \epsilon; 0\ \epsilon\ 1),\ \ |\epsilon| < 1$$

   (e) Find a way to establish the tighter bound $|\lambda_3 - 1| \leq \epsilon^2$ on the smallest eigenvalue of A. (Hint: consider a diagonal similarity transformation)

   (f) Explain why Gershgorin's theorem implies that $A = (1\ 0\ -2\ 0; 0\ 12\ 0\ -4; 1\ 0\ -1\ 0; 0\ 5\ 0\ 0)$ has at least 2 real roots.

4. Consider a matrix $A \in \mathbf{C}^{n\times n}$.

   (a) Define the spectral radius, $\rho(A)$ and the matrix norms $||A||_k$, $k = 1, 2, \infty$.

   (b) Show that $\rho(A) \leq ||A||$ where $||...||$ is any matrix norm induced by a vector norm.

   (c) Show that

$$||A||_2^2 = \rho(A^*A) = \sigma_1^2$$

   where σ_1 is the largest singular value of A.

   (d) Show that

$$||A||_2^2 \leq ||A||_1 ||A||_\infty\ .$$