Numerical Analysis Fall 2005
MS/PhD Qualifying Examination

Instructions: Complete all problems. Explain your answers.

1. (25 points)
   (a) Show that $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ defines an inner product on the space of continuous functions on $[-1,1]$, $C^0_{[-1,1]}$. What is the associated norm?

   (b) State the (classical) Gram-Schmidt orthogonalization (GS) algorithm.

   (c) Apply the GS algorithm to orthogonalize the vectors $\{1, x, x^2\} \in C^0_{[-1,1]}$, using the inner product stated in (a).

2. (25 points)
   (a) Define the Krylov subspace $\mathcal{K}_n$ formed from a matrix A and a vector $\mathbf{b}$, where $A \in \mathbb{C}^{m\times m}$ and $\mathbf{b} \in \mathbb{C}^{m\times 1}$.

   (b) Let A be Hermitian. What is the dimension of $\mathcal{K}_n$ and how does it depend on $\mathbf{b}$? Explain clearly.

   (c) The Conjugate Gradient (CG) algorithm to solve $A\mathbf{x} = \mathbf{b}$ is given by

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$\mathbf{x}_0 = \mathbf{0}$, $\mathbf{r}_0 = \mathbf{b}$, $\mathbf{p}_0 = \mathbf{r}_0$
for $n = 1, 2, 3, \ldots$
    $\alpha_n = (\mathbf{r}_{n-1}^T \mathbf{r}_{n-1})/(\mathbf{p}_{n-1}^T A \mathbf{p}_{n-1})$
    $\mathbf{x}_n = \mathbf{x}_{n-1} + \alpha_n \mathbf{p}_{n-1}$
    $\mathbf{r}_n = \mathbf{r}_{n-1} - \alpha_n A \mathbf{p}_{n-1}$
    $\beta_n = (\mathbf{r}_n^T \mathbf{r}_n)/(\mathbf{r}_{n-1}^T \mathbf{r}_{n-1})$
    $\mathbf{p}_n = \mathbf{r}_n + \beta_n \mathbf{p}_{n-1}$
end
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   Show that $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$ is the residual at the kth step.
   Show that if A is Hermitian, the search directions are A-conjugate, $\mathbf{p}_j^T A \mathbf{p}_k = 0, j \neq k$ and the residuals are orthogonal, $\mathbf{r}_j^T \mathbf{r}_k = 0, j \neq k$.
   Show that $\{\mathbf{r}_k\}_{k=0}^{n-1}$ is an orthogonal basis for the Krylov subspace $\mathcal{K}_n$ formed from A and $\mathbf{b}$.

   (d) In exact arithmetic, how many steps does the CG algorithm require to converge to the exact solution of $A\mathbf{x} = \mathbf{b}$? Explain clearly.

   (e) Let $A \in \mathbb{C}^{m\times m}$, $m = 20$, be given by

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

   The eigenvectors $\mathbf{u}^k$, $k = 1, \ldots, m$ of the matrix A have components $u_j^k = \sin(2\pi k j/m)$, $j = 1, \ldots, m$. The corresponding eigenvalues are $\lambda_k = 2(1 - \cos\frac{k\pi}{m})$. In exact arithmetic, how many steps does the CG algorithm require to converge to the exact solution of $A\mathbf{x} = \mathbf{b}$, where $b_j = \sin(2\pi j/m) - \sin(8\pi j/m)$, $j = 1, \ldots, m$? Explain your answer.

3. (30 points) Let A be an $n \times n$ matrix whose singular values satisfy the inequalities:

$$4^{-k} \leq \sigma_k \leq 4^{1-k} \ .$$

   (a) Derive upper and lower bounds for the 2-norm condition number of A as functions of n.

   (b) Suppose $A\mathbf{x} = \mathbf{b}$ and $\|\mathbf{b}\|_2 = 1$. Derive upper and lower bounds on $\|\mathbf{x}\|_2$.

   (c) Using standard double precision arithmetic, for which $\epsilon_{\text{machine}} \approx 2^{-52}$, roughly for n how large can one expect to solve this linear system for $\mathbf{x}$ with at least two binary digits of accuracy? Assume a stable factorization algorithm.

   (d) Write $\mathbf{x}$ as an expansion in the right singular vectors of A. Assuming the inner product of $\mathbf{b}$ and the kth left singular vector satisfies:

$$|\mathbf{u}_k^T \mathbf{b}| \leq \sigma_k^2,$$

   derive an upper bound on $\|\mathbf{x}\|_2$.

   (e) A method for approximately solving an ill-conditioned system such as the one described above is Tikhonov regularization. The method consists of choosing a regularization parameter, λ, and solving

$$\left(A^T A + \lambda I\right) \mathbf{x}_\lambda = A^T \mathbf{b}.$$

   Write $\mathbf{x}_\lambda$ as an expansion in the right singular vectors of A.

   (f) Making the assumptions of the previous parts (c,d) and assuming that you use a stable factorization algorithm, roughly how should λ be chosen, for given n, to produce the most accurate approximation to $\mathbf{x}$? About how accurate an answer do you expect? (You should assume $n \gg 1$ and you can make the ansatz $\sigma_n \ll \lambda \ll \sigma_1$.)

4. (20 points) Consider the problem of finding approximate solutions to the system of p ordinary differential equations:

$$\frac{dy}{dt} = A(t)y(t) + f(t), \quad 0 \leq t \leq 1,$$

$$A(1+t) = A(t), \quad f(1+t) = f(t), \quad y(1+t) = y(t)$$

with the box scheme. That is, setting $t_j = j/N$, $j = 0, \ldots, N$, introduce $v_j \approx y(t_j)$ satisfying:

$$v_{j+1} - v_j - \frac{1}{2N} A(t_{j+\frac{1}{2}})\,(v_j + v_{j+1}) = \frac{1}{N} f(t_{j+\frac{1}{2}}), \quad j = 0, \ldots, N-1,$$

with $v_0 = v_N$ and defining $(t_{j+\frac{1}{2}}) = (t_j + t_{j+1})/2$.

   (a) Describe the coefficient matrix of the linear system.

   (b) Describe in detail an efficient algorithm for solving this system. Give an operation count to leading order in N large.

   (c) Is the system always solvable for N large? (Consider the approximate system obtained by ignoring the $O(1/N)$ terms.)