Numerical Analysis Exam - Spring 2003

Answer all four questions

1. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is $k \times k$ and nonsingular. Then $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the Schur complement of A_{11} in A.

i. Show that after k steps of Gaussian elimination without pivoting A_{22} has been overwritten with S. (You may assume that no zero pivots are encountered.)

ii. Suppose $A = A^T$, A_{11} is positive definite, and A_{22} is negative definite. Show that A is nonsingular and Gaussian elimination without pivoting works in exact arithmetic.

iii. Construct a 2×2 example satisfying the assumptions of part (ii.) for which Gaussian elimination without pivoting is numerically unstable. That is, find an example where the computed factors using floating point arithmetic, $\hat{L}$, $\hat{U}$, satisfy $\hat{L}\hat{U} = A + E$ where $\|E\|$ is *not small* compared with $\|A\|$ in some standard matrix norm.

2. An $n \times n$ matrix A is skew-Hermitian if $a_{ij} = -a_{ji}^*$, $1 \leq i, j \leq n$. Show that if A is skew-Hermitian then the matrix $U = (I + A)^{-1}(I - A)$ is unitary. Show that any unitary matrix U can be written in the above form for an appropriate skew-Hermitian matrix A, provided the spectrum of U does not contain the eigenvalue $\lambda = -1$.

3. Prove the Schur theorem and show that the Schur factorization of a normal matrix is diagonal. That is:

i. Prove that any square matrix B has the Schur factorization:

$$B = U^H T U$$

where U is unitary and T upper triangular.

ii. Prove that if B is normal, i.e. $BB^H = B^H B$ then B has the factorization

$$B = U^H D U$$

with U unitary and D diagonal.

4. Let A be a square matrix and $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$ be its singular values.

i. Let λ be an eigenvalue of A. Show that $|\lambda| \leq \sigma_1$.

ii. Show that $|det(A)| = \prod_{i=1}^{n} \sigma_i$.