Numerical Analysis Qualifying Exam For August 2003

Instruction: Complete all four problems.

1. **a.** Let A and B be real $m \times n$ matrices. What can you say, in general, about the relationship of the three numbers
$$rank(A), \quad rank(B), \quad rank(A+B) \text{ ?}$$
   b. Let A be a real $m \times n$ matrix. What can you say, in general, about the relationship of the three numbers
$$rank(A), \quad rank(A^T A), \quad rank(AA^T) \text{ ?}$$
   Prove your claims.

2. **a.** Let A be a real $m \times n$ matrix. Define the singular value decomposition of A.
   b. Suppose A is a 7×10 matrix with singular values
$$\sigma_1 = 10, \quad \sigma_2 = 5, \quad \sigma_3 = 1 \text{ ,}$$
   and all other singular values of A are zero.
   What is the rank of A? What is the distance of A to the nearest matrix of rank 2? What is the distance of A to the zero matrix? Explain which distance function you use and justify your answers.

3. Given n real numbers r_j, $j = 1, \ldots, n$, we define an $n \times n$ circulant matrix, C, by:
$$c_{ij} = r_{j-i+1}, \quad r_p \equiv r_{p+n}, \quad p \leq 0.$$
   That is:
$$C = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ r_n & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ r_2 & r_3 & \cdots & r_1 \end{pmatrix}.$$
   a. Show that the discrete Fourier vectors, $q^{(k)}$, defined by:
$$q_j^{(k)} = e^{2\pi i (j-1)(k-1)/n}, \quad j, k = 1, \ldots, n,$$
   are eigenvectors of C. What are the corresponding eigenvalues?
   b. Sketch a fast algorithm for computing Cx and $C^{-1}x$ for favorable values of n. What is the approximate operation count of this fast algorithm?
   c. Let:
$$B = C + uv^T,$$
   where $u, v \in R^n$. Describe a fast algorithm for solving $Bx = y$.

4. Describe three basic algorithms for finding the least squares solution of an overdetermined system of m linear equations in n unknowns. How do they compare in terms of cost? Discuss their accuracy.