

Numerical Analysis Spring 2006
MS/PhD Qualifying Examination

Instructions: Write the last four digits of your SSN (not your name) on each sheet. Complete all problems. Clear and concise answers with good justification will improve your score.

1. (25 points) This question concerns properties of the null space $N(A)$ and the range space $R(A)$ of a complex-valued matrix A and its complex conjugate transpose A^* .

(a) Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix}$$

This matrix gives a mapping of \mathbb{R}^n into \mathbb{R}^m . What are n and m ? Put the matrix A in **reduced** row echelon form. Find the $N(A)$, $N(A^*)$, $R(A)$, and the $R(A)$.

- (b) Now suppose that A is a linear transformation mapping \mathbb{C}^n into \mathbb{C}^m for some $n, m > 0$. Prove that the range of A is perpendicular to the null space of A^* : $N(A) \perp R(A^*)$. Also show that the \mathbb{R}^n is the direct sum of $N(A)$ and $R(A^*)$: $\mathbb{R}^n = N(A) \oplus R(A^*)$.

- (c) The fundamental theorem of linear algebra is often stated in the form of Fredholm's alternative: For any A and b , one and only one of the following systems has a solution:

$$Ax = b, \quad \text{or} \quad A^*y = 0, \quad y^*b \neq 0.$$

Prove this statement.

2. (25 points) This problem concerns the conditioning or stability of eigenvalues and eigenvectors. Let $A(\epsilon) = A + \epsilon B$ where A and B are n by n complex-valued matrices and $\lambda(\epsilon)$, $v(\epsilon)$ be and eigenpair for $A(\epsilon)$:

$$A(\epsilon)v(\epsilon) = \lambda(\epsilon)v(\epsilon).$$

The stability of the eigenvalue $\lambda(0)$ of A and eigenvector $v(0)$ of A are measured by

$$\frac{d\lambda}{d\epsilon}(0), \quad \frac{dv}{d\epsilon}(0).$$

- (a) Determine the stability of the eigenpairs of A for

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- (b) Determine the stability of the eigenpairs of A for

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- (c) Now assume that A and B are n by n matrices and that $\lambda(0)$ is a simple eigenvalue of A . You may assume, as is known to be true, that for small ϵ there exist $\lambda(\epsilon)$, $v(\epsilon)$ and $w(\epsilon)$ that are differentiable functions of ϵ and

$$A(\epsilon)v(\epsilon) = \lambda(\epsilon)v(\epsilon), \quad w(\epsilon)^*A(\epsilon) = \lambda(\epsilon)w(\epsilon)^*.$$

Show that the eigenvalue is well conditioned (stable). State your answer in terms of the cosine of the angle between $v(0)$ and $w(0)$.

3. (25 points)

- Define Householder reflections or Givens rotations. Show that they are orthogonal.
- Describe in detail an algorithm using whichever transformations you discussed in part (a) to compute the QR factorization of an $m \times n$ matrix, A . How many flops are required to compute the factorization?
- Discuss the backward stability of the factorization algorithm.
- Assuming $m > n$, describe an algorithm using the factorization to compute the least squares solution to $Ax = b$. Show that your algorithm indeed finds the least squares solution in exact arithmetic. How many flops does it require in addition to those needed to compute Q and R ?

4. (25 points) Consider the solution of the square system of linear equations:

$$Ax = b$$

by the iterative method:

$$\begin{aligned} r^{(k)} &= b - Ax^{(k)}, \\ Pz^{(k)} &= r^{(k)}, \\ x^{(k+1)} &= x^{(k)} + z^{(k)}. \end{aligned} \tag{1}$$

Here the matrix P is typically called a preconditioner.

- Assuming P is invertible, show that if the sequence of iterates converges then it converges to a solution of the original system.
- Give necessary and sufficient conditions on P and A for the convergence of the method with an arbitrary initial guess, $x^{(0)}$.
- Show that the iterates are of the form:

$$x^{(k)} = x^{(0)} + s^{(k)},$$

where $s^{(k)}$ is in the Krylov subspace:

$$s^{(k)} \in \mathcal{K}_{k+1}(P^{-1}A; P^{-1}r^{(0)}) \equiv \text{span} \left\{ P^{-1}r^{(0)}, P^{-1}AP^{-1}r^{(0)}, \dots, (P^{-1}A)^k P^{-1}r^{(0)} \right\}.$$

- Describe in detail an algorithm for accelerating the convergence of the iterations by replacing $s^{(k)}$ by a vector $\tilde{s}^{(k)} \in \mathcal{K}_{k+1}(P^{-1}A; P^{-1}r^{(0)})$ chosen so that the residual is minimized. (The algorithm is called GMRES.) Discuss how efficiency is achieved by updating an orthonormal basis of the Krylov subspace.