

Numerical Analysis Fall 2001
MS/PhD Qualifying Examination

Instruction: Complete all four problems.

1. Let \mathcal{S} denote the subspace of \mathbb{R}^3 consisting of all vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with

$$x_1 + x_2 = 3x_3 .$$

- a** Find an orthonormal basis for \mathcal{S} .
- b** What is the dimension of \mathcal{S}^\perp , the orthogonal complement of \mathcal{S} ? Find a basis for \mathcal{S}^\perp .
- c** Write the vector

$$b = \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}$$

as $b = u + v$ with $u \in \mathcal{S}$ and $v \in \mathcal{S}^\perp$.

2. **a** Given n linearly independent vectors in \mathbb{R}^m , describe the Gram–Schmidt orthogonalization process.
- b** Given a matrix $A \in \mathbb{R}^{m \times n}$ with linearly independent columns. Show that one can factorize $A = QR$ where Q has orthonormal columns and R is upper triangular.
- c** Explain the relation between the Gram–Schmidt process and the factorization $A = QR$.
- d** Given an overdetermined linear system $Ax = b$ with A as in b). Explain how the factorization $A = QR$ can be used to determine the least–squares solution.

3. Let H be a nonsingular, upper Hessenberg matrix. Describe the application of Gaussian elimination with partial pivoting to this matrix. Describe the structure of the factors computed. Estimate the number of flops required to compute the factorization. (Recall that H is upper Hessenberg if $h_{ij} = 0$ whenever $i > j + 1$.)
4. Let A be a real, symmetric matrix.
- a** Show that all eigenvalues, λ , of A are real and that A can be diagonalized by an orthogonal similarity transformation.
- b** Given a nonzero vector, v , the Rayleigh quotient, $R_A(v)$, is defined by:

$$R_A(v) = \frac{v^T A v}{v^T v}.$$

Show that that v is a stationary point of R_A - that is $\nabla_v R_A = 0$, if and only if v is an eigenvector of A . If v is an eigenvector, what is the value of $R_A(v)$?

- c** Rayleigh quotient iteration is defined by:

$$y^{(n+1)} = \left(A - R_A(x^{(n)})I \right)^{-1} x^{(n)},$$

$$x^{(n+1)} = \frac{y^{(n+1)}}{\|y^{(n+1)}\|_2}.$$

Prove that Rayleigh quotient iteration is locally cubically convergent to simple eigenvalue-eigenvector pairs.