Numerical Analysis Fall 1998
MS/PhD Qualifying Examination

Instructions: Complete all four problems.

1. (a) Prove that an $m \times n$ matrix has $\text{rank}(\mathbf{A}) = 1$ if and only if $\mathbf{A} = \vec{x}\vec{y}^T$ for some $\vec{x} \in \mathbf{R}^m$, $\vec{y} \in \mathbf{R}^n$.

   (b) Let $\mathbf{A}$ be $m \times n$ and $\mathbf{B}$ be $n \times m$ with $\text{rank}(\mathbf{B}) = m \leq n$. Prove or disprove (by counterexample) the assertions that:

$$\begin{aligned} \text{rank}(\mathbf{AB}) &= \text{rank}\mathbf{A} \\ \text{rank}(\mathbf{BA}) &= \text{rank}\mathbf{A} \end{aligned}$$

2. (a) Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. Show that $\lambda = 1$ is a double eigenvalue, but that there is only one eigenvector, $\mathbf{e}$; *i.e.* $\mathbf{Ae} = \lambda\mathbf{e}$. Find a vector $\mathbf{v}$ such that $\mathbf{Av} = \lambda\mathbf{v} + \mathbf{e}$. Thus, show that $\mathbf{A}(\mathbf{e} \;\; \mathbf{v}) = (\mathbf{e} \;\; \mathbf{v}) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

   (b) Given the recurrence relation

$$F_{k+2} = 2F_{k+1} - F_k, \quad F_0 = 0, F_1 = 1;$$

   find F_k and the limit, $\lim_{k\to\infty} F_{k+1}/F_k$, if it exists.

3. The preliminary reduction of a symmetric matrix to tridiagonal form before applying the QR-algorithm for finding eigenvalues and eigenvectors would be of little use if the steps in the QR-algorithm did not preserve this structure.

   (a) In the QR-factorization, $\mathbf{A} = \mathbf{QR}$, of a tridiagonal matrix $\mathbf{A}$, which entries of $\mathbf{R}$ are generally nonzero? Which entries of $\mathbf{Q}$? (In practice, $\mathbf{Q}$ is not formed explicitly.)

   (b) Show that the tridiagonal structure is recovered when the product $\mathbf{RQ}$ is formed.

   (c) Explain how Givens rotations or 2x2 Householder reflections can be used in the QR-factorization of a tridiagonal matrix, reducing the operation count far below what would be required for a full matrix. Compare operation counts for the tridiagonal and full case.

4. (a) Give a precise statement of the Gaussian elimination algorithm without pivoting for reducing a square matrix to upper triangular form.

   (b) Show that if $\mathbf{A}$ is a symmetric, positive definite matrix, then the algorithm in (a) cannot fail, *i.e.*, all the pivots are nonzero.