

Numerical Analysis
MS/PhD Qualifying Examination
Fall 1997

Instructions: Complete all four problems.

1. What is the dimension of the set of vectors $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ which are simultaneously solutions

of

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and of

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}?$$

Find a basis for the solution set.

2. Consider the family of square matrices $\mathbf{A}(\theta)$ defined by the solution of the differential equation

$$\frac{d\mathbf{A}(\theta)}{d\theta} = \mathbf{B}\mathbf{A}(\theta),$$

with initial condition $\mathbf{A}(0) = \mathbf{I}$ where \mathbf{B} is a constant square matrix of the same order as \mathbf{A} .

- (i) Find necessary and sufficient conditions on \mathbf{B} which make $\mathbf{A}(\theta)$ orthogonal for all θ .

- (ii) Find the matrices $\mathbf{A}(\theta)$ corresponding to $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. (i) Suppose \mathbf{A} is an invertible $n \times n$ matrix and \mathbf{U}, \mathbf{V} are $n \times k$ matrices such that $\mathbf{A} + \mathbf{UV}^T$ is also invertible. Prove the Sherman-Morrison-Woodbury formula:

$$(\mathbf{A} + \mathbf{UV}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{I} + \mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^T\mathbf{A}^{-1}.$$

- (ii) Let \mathbf{T} be a $2m \times 2m$ tridiagonal matrix and let \mathbf{A} be obtained from \mathbf{T} by setting the elements $t_{m,m+1}$ and $t_{m+1,m}$ to zero. Show how to use the Sherman-Morrison-Woodbury formula to express \mathbf{T}^{-1} in terms of \mathbf{A}^{-1} .
- (iii) Compare the number of arithmetic operations required to solve $\mathbf{T}\mathbf{x} = \mathbf{y}$ using the method of part (ii) to the number required by the standard banded Gaussian elimination algorithm.
- (iv) Consider the generalization of this algorithm to the decomposition of \mathbf{T} into p pieces of size m . How does the total work scale with p and $n = mp$? Can you think of any circumstances where this algorithm could be more efficient than the standard banded elimination algorithm?
4. (i) Define the power method for computing the eigenvalue(s) of a real matrix, \mathbf{A} , with maximum modulus. How does the algorithm behave when the eigenvalues with maximum modulus are a complex conjugate pair?
- (ii) Show that for τ real, positive, and sufficiently small, the eigenvalue(s) of the matrix $\mathbf{I} + \tau\mathbf{A}$ with maximum modulus corresponds to the eigenvalue(s) with largest real part.
- (iii) Describe an algorithm based on the power method to compute the eigenvalue of \mathbf{A} with largest real part.