Numerical Analysis Exam
January 13, 199~~4~~5

Complete all four problems. Unless otherwise stated, all matrices and vectors are real. Good luck!

1. **a.** Show that for any nonsingular $n \times n$ matrix S, the $n \times n$ matrices A and SAS^{-1} have the same eigenvalues.

   b. Show that all eigenvalues of the matrix A with elements,
$$a_{ij} = i^2(i-j)^3 j^4, \quad i,j = 1,\ldots,n,$$
   have zero real part.

2. Consider the symmetric matrix, M, partitioned into blocks as follows:
$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$
   where A and C are square matrices, not necessarily of the same size, and where B is a rectangular matrix with transpose B^T. Show that, if M is positive definite, then the following matrices are positive definite:

   (i.) A

   (ii.) $C - B^T A^{-1} B$.

3. Let M be a positive definite symmetric matrix.

   a. Show that if Gaussian elimination is applied to M, then only positive pivots are encountered.

   b. Use the result above to deduce the existence of the Cholesky factorization, that is a lower triangular matrix, L, such that $M = LL^T$.

   c. Describe an algorithm for computing L which requires about half the work as standard Gaussian elimination.

4. Consider the matrix $H = I - 2uu^T$ where $u^T u = 1$.

   a. Prove that H is an orthogonal matrix.

   b. Let
$$u = \frac{1}{\|a + \sigma e_1\|}(a + \sigma e_1),$$
   where $\sigma = \pm\|a\|$, $e_1^T = (1, 0, \ldots, 0)$, and $\|\cdot\|$ is the 2-norm. Compute Ha.

   c. Describe an algorithm which uses matrices of the form H to reduce an $n \times n$ matrix A to upper triangular form.