Instructions: Do the following 7 problems. Show all your work. Notice the indicated relative value of each problem.

(1) (10 points) Find an open covering of the set $S = \{1/n : n = 1, 2, \ldots \} \subset \mathbb{R}$ that does not contain a finite subcovering. Is it possible to do the same for the set $S \cup \{0\}$?

(2) (20 points) On the unit sphere $S^n$, define the relation that identifies a point $x$ with its antipodal $-x$. Let $\mathbb{R}P^n$ be the quotient space. Prove that $\mathbb{R}P^n$ is a Hausdorff space.

(3) (10 points) Compute the fundamental group of the doubly punctured sphere $S^2 \setminus \{p, q\}$. Here $p$ and $q$ are any two given points on $S^2$.

(4) (20 points) Let $X$ be a path-connected topological space such that $\pi_1(X, x_0)$ is a finite group. Prove that any continuous function $f : X \to \mathbb{S}^1 \times \mathbb{S}^1$ is homotopic to a constant map.

(5) (20 points) Let $G$ be a topological group, $X$ a Hausdorff space, and $G \times X \to X$ a continuous action of $G$ on $X$. Let $I(x) = \{g \in G : gx = x\}$ and $O(x) = \{y \in X : y = gx$ for some $g \in G\}$.
   (a) Show that $I(x)$ is a closed subgroup of $G$.
   (b) Show that if $y \in O(x)$, then $I(x)$ and $I(y)$ are conjugate subgroups of $G$.
   (c) Show that if $G$ is compact, then $G/I(x)$ is homeomorphic to $O(x)$.

(6) (20 points) Let $\mathbb{GL}(n)$ denote the Lie group of non-singular $n \times n$ matrices, and let $\mathbb{SL}(n) = \{A \in \mathbb{GL}(n) : \det A = 1\}$.
   (a) Show that $\mathbb{SL}(n)$ is a Lie group.
   (b) Show that $\mathbb{SL}(n)$ is connected and noncompact.
   **Hint** (for connectedness): use elementary matrices to show that $\mathbb{SL}(n)$ is path-connected.

(7) (20 points) Let $f : M^{(n)} \to N^{(k)}$ be a smooth map between differentiable manifolds. Suppose that $df(x) : T_xM \to T_pN$ is surjective for all $x \in f^{-1}(p)$. Show that $f^{-1}(p)$ is an $n-k$ dimensional submanifold of $M$. 
**Hint:** use the fact that if \( g : \mathbb{R}^n \to \mathbb{R}^k \) is a smooth map such that \( g(0) = 0 \) and \( \text{D}g(0) \) has rank \( k \), then there exists a local diffeomorphism \( \varphi : U \ni 0 \to \varphi(U) \subset \mathbb{R} \) such that \( g \circ \varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_k) \).