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COMPLEX ANALYSIS QUALIFYING EXAMINATION

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DIRECTIONS: You are trying to convince the reader that you know what your are doing, so you should give clear, concise and complete answers explaining your work. Note that holomorphic is synonymous with analytic. **Do any 7 of the following 10 problems.**

1. Compute the radius of convergence of the following:

a) $\displaystyle\sum_{1}^{\infty} \frac{(3n)!}{(3n)^{3n}} z^n.$

b) $\displaystyle\sum_{0}^{\infty} [3+(-1)^n]^n z^n.$

c) The Taylor series around zero for the function $\dfrac{z}{e^z-1}$.

2. Suppose that the linear transformation $T(z) = \dfrac{az+\cancel{d}\;b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ has three fixed points. Prove that T is the identity transformation.

3. Verify that a suitable branch of the function $f(z) = \log(5 + \sqrt{\frac{z+1}{z-1}})$ is a well defined (single-valued) holomorphic function in the z-plane outside of a line joining the points $z = 1$ and $z = -1$. Show, however, that if one enters another sheet of the Riemann surface by crossing this line, there will be a branch point at $z = 13/12$.

4. Let $f(z)$ be a holomorphic function in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$. Prove that

$$f(z) + f(z^2) + f(z^3) + \cdots + f(z^n) + \cdots$$

converges uniformily on compact subsets of D.

5. Let $f(z)$ and $g(z)$ be entire functions. Show that if $f \circ g(z)$ is a polynomial then both $f(z)$ and $g(z)$ are polynomials.

6. Let $f(z)$ be a holomorphic function on the closed unit disc $\bar{D} = \{z \in \mathbb{C} : |z| \le 1\}$ that satisfies $|f(z)| < 1$ for $|z| = 1$. Prove that f has exactly one fixed point in the open unit disc D.

7. Define the function

$$F(z) = \int_{|\zeta|=2} \frac{d\zeta}{\zeta(\zeta - z)(\zeta - z + 1)}$$

Determine the limit of $F(z)$ as $z \to 2$ from

1) inside the circle $|z| = 2$,

2) outside the the circle $|z| = 2$.

3) Is $F(z)$ continuous at $z = 2$?

8. Show by the method of complex contour integration that the following identities hold:

a)

$$\int_0^\infty \frac{\sin ax}{x(x^2+1)} dx = \frac{1 - e^{-a}}{2}.$$

b)

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \pi.$$

9. Expand $e^{1/z}$ in a Laurent series about $z = 0$, determining the Laurent coefficients. Show that

$$\frac{1}{n!} = \frac{1}{\pi} \int_0^\pi e^{\cos\theta} \cos(n\theta - \sin\theta) d\theta.$$

10. Construct a meromorphic function that has simple poles at the non-zero integers with residue at $z = n$ equal to n^2.