

Complex Analysis Qualifying Exam
January 16, 2003

Instructions: *There are nine problems, please do eight of them. Start each problem on a new sheet of paper and write on one side of each sheet of paper. Remember to write the last four digits of your Social Security number in all pages and to clearly number them. Good luck!!*

1. (a) Let the roots of $(z - 1)^n + z^n = 0$ be denoted by z_k , $k = 0, 1, \dots, n - 1$. Show that all the roots z_k lie on the line $x = 1/2$.

(b) Let w be any n -th root of the unity not equal to 1. Find the sum

$$1 + 2w + 3w^2 + 4w^3 \dots + nw^{n-1}.$$

2. (a) Show that the only conformal maps from the complex plane onto itself (bijections!) are the non-constant linear maps, i.e. maps of the form $f(z) = az + b$, $a \neq 0$.

(b) Show that the only conformal maps from the unit disc onto itself (bijections!) are the Moebius transformations of the form

$$Tz = e^{i\alpha} \left(\frac{z - a}{1 - \bar{a}z} \right),$$

where $\alpha \in \mathbf{R}$ and $|a| < 1$.

3. (a) Classify the singularities of the function

$$f(z) = \frac{z}{\sin z}.$$

Include the point at infinity.

(b) Find a Laurent expansion, valid in the region $|z + 1| > 3$, for

$$f(z) = \frac{7z - 2}{z^3 - z^2 - 2z}.$$

Find the residue of $f(z)$ at $z = 0$.

4. Find the sum of the series

$$f(z) = \sum_{n=0}^{\infty} e^{-n} \sin(nz)$$

and indicate the domain of convergence. Find the domain of analyticity of $f(z)$, and calculate $f'(z)$.

5. Let f be analytic in \mathbf{C} . Assume $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed constant $M > 0$, and a sequence of values of r going to infinity. Show that f is a polynomial of degree less than or equal to n .

6. Evaluate the following integrals:

(a) $\oint_{\gamma} \frac{e^z}{(z+1)(z-2i+1)} dz,$

where γ is the ellipse given by $\frac{x^2}{4} + y^2 = 1$, with positive orientation (counterclockwise!).

(b) $\int_0^{\infty} \frac{\ln x}{4+x^2} dx.$

7. Give two distinct harmonic functions on \mathbf{C} that vanish on the entire real axis. Why is this not possible for analytic functions?

8. Prove that if f is analytic on the region U (open and simply connected), $z_0 \in U$, and $f'(z_0) = 0$, then f is not one-to-one in any neighborhood of z_0 .

9. (a) State the Mittag-Leffler Theorem.

(b) Show that

$$\pi \tan(\pi z) = \sum_{n=1}^{\infty} \frac{2z}{\left(\frac{2n-1}{2}\right)^2 - z^2}.$$